Electricity and Magnetism
Version A: Full Math

W. Blaine Dowler

April 21, 2012

Contents

1 Electricity and Magnetism the Newtonian Way

We have already seen how the laws of electricity and magnetism led to the rise of relativity. Let us examine how relativity alters them more closely.

Both electrical and magnetic phenomena have been studied for thousands of years. It was not until 1820, however, that the scientific community at large recognized how closely connected the two are. Hans Christian Oersted noticed (which giving a lecture to students) that electric currents deflect the needle of a compass. Further studies were able to quantify the relationship, explicitly defining and mathematically defining the procedure. It was clear that an electrically charged object in motion produces a magnetic field that is circular in shape and surrounds the particle at a right angle to the direction of motion. There was a problem, though: there was no *theoretical* reason to connect the two interactions. The two types of phenomena were kludged together mathematically because the connection was experimentally proven, and *not* because of any intrinsic motivation to tie the two together. It was comparable to rain and rainbows: people recognized that rainbows appeared after rainfalls so early on that even the word "rainbow" includes the word "rain," but it would be centuries before the optics involved were understood well enough to explain precisely why rainbows appeared.

The theoretical descriptions of electrical and magnetic phenomena was completed in the late 1860s, and James Clerk Maxwell collated the results into what are now known as Maxwell's equations. These described all known electrical and magnetic phenomena, detailing the connections between the fields without explaining why they are connected, with particular focus and attention on how the electric and magnetic fields change and evolve through space and time.[1] Just as ancient people could predict the appearance of rainbows, people in the 1900s could predict the magnetic fields produced by electrically charged objects to great effect, but the connection was arbitrary. This was still a young science, so when it was shown to be inconsistent with Newtonian physics, it was assumed that the fault lay with electricity and magnetism. As both theories were consistent with the experimental data available at the time, this assumption seemed reasonable. As we learned in earlier lessons, it was actually the Newtonian mechanics that were at fault.

---

[1]Remember, to James Clerk Maxwell and his contemporaries, space and time were two completely distinct quantities.

2 Electricity and Magnetism the Relativistic Way

The work of Lorentz and Einstein combined to solve the problems with electricity and magnetism in multiple frames of reference. Furthermore, they showed that electromagnetic theory did not need to be altered in any special way; the framework established by Maxwell's equations was completely consistent with electricity and magnetism. There were, however, other significant benefits to the relativistic framework that were worth exploring.

So far, when we take a relativistic viewpoint of quantities, we have found a common pattern: quantities that depend on direction get a fourth quantity attached which serves as the "time direction" component of the quantity. In all such cases, the fourth quantity was independent of direction. In the case of electrical fields, the prime candidate for such a fourth quantity is the magnetic field, but that *also* depends on direction. It can't fit easily into that fourth slot. This seems to be a stumbling block on the path to connecting electricity and magnetism on the theoretical side.

Undeterred, we go back to what we learned in lesson four: although no *simple* rotation can move something from a spatial direction into the time direction, one can formulate a rotation based on acceleration that does the same job. This becomes the key to connecting the two phenomena.

Imagine, if you will, a long, electrically charge rod that is at rest. In the classical view, accelerating this rod along its axis[2] causes no change in the electrical field, but produces a sudden and inexplicable magnetic field. If we had two such rods lying parallel and moving in identical directions at identical speeds, the magnetic force created would be an attractive force.

We now know that such a motion will increase the electrical force: as the rods accelerate, their lengths contract. With the same amount of charge packed into a shorter distance, we see the electrical charge density increase, which causes an increase in the electrostatic attraction. By carefully calculating the result of "rotating" the electric field to a higher velocity using the math that describes our relativistic world, we find that the action of the classical magnetic field is *identical* to the action of this "extra" electric field produced by relativity! This math is independent of the situation, so it doesn't matter if our moving electrical charge is a rod, disc, point, etc. This is true for *any* shape that moves while carrying an electrical charge.

This was something of a holy grail to physicists: the elusive theoretical connection between the electric and magnetic fields that didn't exist in the classical

---

[2]By "along its axis," we mean that if the rod lies in the North/South direction, then we accelerate it either North or South, and not East, West, up or down.

view became an immediate and natural consequence of relativistic dynamics. The magnetic fields observed around active circuits resulted directly from the motion of charged particles under the rules of relativity. The mathematical object used to combine the two is more complicated than a four directional vector, but it is just as valid: the fields were unified into a single phenomenon, and Maxwell's set of four[3] equations simplified to a single equation. This served as yet another triumph of the special theory of relativity. Despite a rocky start, the scientific community had finally accepted special relativity as the theory that explained reality better than any other theory. It is now known that any errors in relativity are small, as the theory has been tested with incredible experimental rigor. Still, special relativity is limited to points of view that do not experience forces or accelerations. Most of our work is done in the presence of a gravitational force, and many real world observers need to change their direction to do their jobs. The special theory of relativity was inadequate for these tasks, and would need to be expanded into the general theory of relativity, which is the subject of our final lessons.

3   Maxwell's Equations

James Clerk Maxwell compiled all that was known about electricity and magnetism into four equations.[4] Those equations are as follows:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

where

$$\nabla = \left[\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right]$$

is the usual differential operator, $\mathbf{E}$ is the electric field, ϵ_0 is the electric permittivity of space, ρ is the charge density, $\mathbf{B}$ is the magnetic field, t is time, μ_0 is the magnetic permeability of free space and $\mathbf{J}$ is the current density. As Lorentz already showed, these equations were consistent with relativity from the start.

---

[3]Well, technically, Maxwell did his work before physicists adopted vectors as mathematical tools, so his "four" equations were originally published as twelve equations. In a roundabout sort of way, this also explains why the symbols for so many electromagnetic variables are counter intuitive.

[4]Having done his work before vectors were adopted by physicists, he actually had twelve equations.

4 General Mathematical Tools

With the electric and magnetic fields intrinsically connected, a simple four vector will not be enough to describe them. There are at least six components of these fields to deal with; three spatial components of the electric field, and three spatial components of the magnetic field. Before we can effectively formulate what this new tensor would look like, we need to introduce some mathematical shorthand tools.

4.1 The Levi-Civita Symbol

The Levi-Civita symbol is one that is quite useful in vector analysis. It has a minimum of three indices, and takes one of three values:

$$\epsilon_{\mu\nu\alpha} = \begin{cases} 1 & \mu\nu\alpha \text{ form an even permutation of 1, 2, and 3. (e.g. 123, 231, 312)} \\ -1 & \mu\nu\alpha \text{ form an odd permutation of 1, 2, and 3. (e.g. 132, 321, 213)} \\ 0 & \text{otherwise. (e.g. any index repeated)} \end{cases}$$

The Levi-Civita symbol makes cross products a bit easier to manage. Under the normal definition, the cross product $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ is defined as the determinant of the matrix-type object

$$\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{bmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{bmatrix}$$

This has two big disadvantages. From the mathematical perspective, it is putting vectors and scalars together into a "matrix" and taking the determinant, but a matrix requires all of its entries to be of the same type. Thus, it's not technically a permissible mathematical object. From the educational perspective, it is often useful to teach cross products before linear algebra has been taught, resulting in a haphazard teaching method that only allows component wise work when the vectors $\mathbf{A}$ and $\mathbf{B}$ are perfectly perpendicular. There is, however, a certain regularity and a pattern to these subscripts. We can redefine the cross product components as follows, given that $\mathbf{A} \times \mathbf{B} = \mathbf{C}$:

$$C_i = \sum_j \sum_k \epsilon_{ijk} A_j B_k$$

To calculate C_x, for example, we calculate

$$
\begin{aligned}
C_x &= \epsilon_{xxx}A_xB_x + \epsilon_{xxy}A_xB_y + \epsilon_{xxz}A_xB_z \\
&\quad + \epsilon_{xyx}A_yB_x + \epsilon_{xyy}A_yB_y + \epsilon_{xyz}A_yB_z \\
&\quad + \epsilon_{xzx}A_zB_x + \epsilon_{xzy}A_zB_y + \epsilon_{xzz}A_zB_z \\
C_x &= 0 + 0 + 0 \\
&\quad + 0 + 0 + A_yB_z \\
&\quad + 0 - A_zB_y + 0 \\
C_x &= A_yB_z - A_zB_y
\end{aligned}
$$

There are a few more terms involved at first glance, but with a bit of practice this is highly efficient, as the mathematician/physicist/student in question chooses not to write down the terms with repeated indices in the first place. The Levi-Civita symbol easily extends to four dimensions as well, using indices 0, 1, 2, and 3.

4.2 Derivatives with Tensors

When we begin working in electromagnetism, taking derivatives will be an unavoidable situation. We need to be careful, however, since we have both four-vector components and one form components. The four-vector components are unmodified by the metric, but the one form components are, so taking the derivative with respect to each provides different results. We want to differentiate four-vectors with respect to four-vector components and one forms with respect to one form components. Algebraically, we represent this with commas in our subscripts and superscripts.

We define a partial derivative with respect to a four-vector component as

$$
\frac{\partial A^\mu}{\partial x^\nu} = A^\mu{}_{,\nu}
$$

Notice that a raised exponent in the denominator becomes a lowered exponent in the numerator. This facilitates the use of Einstein summation convention. Notice also the comma that is in place to represent the partial derivative.

We have already made use of the ∇ operator for taking derivatives in three dimensions, as defined above. The three sided shape is useful for a three dimensional differential operator. The convention for the four dimensional differential

operator is the natural extension of this:

$$\vec{\Box} = \begin{pmatrix} \frac{\partial}{\partial x^0} \\ \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \end{pmatrix}$$

Thus,

$$\Box^2 = \Box^\mu \Box_\mu = -\frac{\partial^2}{c^2 \partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

where $\Box^\mu$ is the derivative with respect to *one form* components and $\Box_\mu$ is the derivative with respect to *four vector* components. This is somewhat counter intuitive, but it is so defined to facilitate the use of Einstein summation convention in cases such as

$$\vec{\Box} \cdot \vec{A} = \Box_\mu A^\mu$$

4.3 Symmetric and Antisymmetric Tensors

Imagine a 4×4 tensor. Generally speaking, this tensor will have 16 possible components. There are, however, two circumstances in which these can be simplified to either 10 or 6 components.

4.3.1 Symmetric Tensors

Our metric

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and our Lorentz transformation tensor

$$\Lambda^{\mu'}{}_\nu = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are both symmetric tensors. Symmetric tensors are defined by the property

$$S^{\mu\nu} = S^{\nu\mu}$$

indicating a symmetry about the diagonal line that starts in the upper left corner of the tensor and continues to the lower right corner of the tensor. In

7

this case, there are only 10 independent entries. Once S^{01} has been determined, S^{10} follows immediately. In the case of an antisymmetric tensor,

$$A^{\mu\nu} = -A^{\nu\mu}$$

which reduces the situation to only 6 independent entries: S^{01} still determines S^{10} and the other five entries on that side of the diagonal. The entries on the diagonal itself must conform to

$$A^{\mu\mu} = -A^{\mu\mu} = 0$$

so that the diagonal of every antisymmetric tensor is identically 0. This is our best bet for the shape of the object that will be our electromagnetic field tensor, containing information about the three electric field components and the three magnetic field components.

5   The Electromagnetic Field Tensor

The electromagnetic field tensor cannot be explicitly derived from scratch, simply because there is more than one way to formulate it. Instead of deriving them, the two forms will be displayed and shown to behave as required.

The most common form of the electromagnetic field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

The alternate form is known as the dual tensor, and it is

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{pmatrix}$$

The structures are clearly similar between the two forms.

To show that this is useful, we must prove that it transforms and reacts to Lorentz transformations as expected. In tensor notation, the transformation would be written

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu}\Lambda^{\nu'}{}_{\nu}F^{\mu\nu}$$

which requires two instances of the Lorentz transformation tensor to transform the two indices. Let us look at a simplified example: we look at a situation in

which a charged particle is at rest in frame S, producing no magnetic field, and is boosted into the S' frame. If we use only one Lorentz transformation tensor we have

$$
\begin{pmatrix}
\gamma & -\frac{v}{c}\gamma & 0 & 0 \\
-\frac{v}{c}\gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\
-\frac{E_x}{c} & 0 & 0 & 0 \\
-\frac{E_y}{c} & 0 & 0 & 0 \\
-\frac{E_z}{c} & 0 & 0 & 0
\end{pmatrix}
$$
$$
=
\begin{pmatrix}
\frac{v\gamma E_x}{c^2} & \frac{\gamma E_x}{c} & \frac{\gamma E_y}{c} & \frac{\gamma E_z}{c} \\
-\frac{\gamma E_x}{c} & -\frac{v\gamma E_x}{c^2} & -\frac{v\gamma E_y}{c^2} & -\frac{v\gamma E_z}{c^2} \\
-\frac{\gamma E_y}{c} & 0 & 0 & 0 \\
-\frac{\gamma E_z}{c} & 0 & 0 & 0
\end{pmatrix}
$$

which lacks the antisymmetric property we require. This is because we have only transformed one of the two indices; we have transformed columns but not rows, so that the components have not been transformed completely. Transforming the rows (by placing the matrix Λ^T on the right and multiplying) results in

$$
\begin{pmatrix}
\frac{v\gamma E_x}{c^2} & \frac{\gamma E_x}{c} & \frac{\gamma E_y}{c} & \frac{\gamma E_z}{c} \\
-\frac{\gamma E_x}{c} & -\frac{v\gamma E_x}{c^2} & -\frac{v\gamma E_y}{c^2} & -\frac{v\gamma E_z}{c^2} \\
-\frac{\gamma E_y}{c} & 0 & 0 & 0 \\
-\frac{\gamma E_z}{c} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\gamma & -\frac{v}{c}\gamma & 0 & 0 \\
-\frac{v}{c}\gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$
$$
=
\begin{pmatrix}
0 & \frac{E_x}{c} & \frac{\gamma E_y}{c} & \frac{\gamma E_z}{c} \\
-\frac{E_x}{c} & 0 & -\frac{v\gamma E_y}{c^2} & -\frac{v\gamma E_z}{c^2} \\
-\frac{\gamma E_y}{c} & \frac{v\gamma E_y}{c^2} & 0 & 0 \\
-\frac{\gamma E_z}{c} & \frac{v\gamma E_z}{c^2} & 0 & 0
\end{pmatrix}
$$

which is the antisymmetric tensor with the correctly transformed components:

$$
\mathbf{E'} =
\begin{bmatrix}
E_x \\
\gamma E_y \\
\gamma E_z
\end{bmatrix}
$$

and

$$
\mathbf{B'} =
\begin{bmatrix}
0 \\
\frac{v\gamma E_z}{c^2} \\
-\frac{v\gamma E_y}{c^2}
\end{bmatrix}
$$

6 Maxwell's Equations Redux

In their relativistic form, Maxwell's equations in free space (in which ρ and $\mathbf{J}$ are both zero) can be reduced to a single equation by using the electromagnetic field tensor and its derivatives.

We begin by examining the derivatives of the electromagnetic field tensor. Starting with the derivatives of the first row, we get

$$
\begin{aligned}
F^{0\nu}{}_{,\nu} &= \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} \\
&= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \\
&= \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho
\end{aligned}
$$

where we have the first of Maxwell's equations, presented above as equation 1 on page 4, in the final step.

Examination of the second row gives us

$$
\begin{aligned}
F^{1\nu}{}_{,\nu} &= \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} \\
&= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + 0 + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\
&= \mu_0 J_x
\end{aligned}
$$

where we have applied a slightly manipulated version of equation 4 on page 4 in the final step. Similarly,

$$
\begin{aligned}
F^{2\nu}{}_{,\nu} &= \frac{\partial F^{20}}{\partial x^0} + \frac{\partial F^{21}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{23}}{\partial x^3} \\
&= -\frac{1}{c^2}\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + 0 + \frac{\partial B_x}{\partial z} \\
&= \mu_0 J_y
\end{aligned}
$$

and

$$
\begin{aligned}
F^{3\nu}{}_{,\nu} &= \frac{\partial F^{30}}{\partial x^0} + \frac{\partial F^{31}}{\partial x^1} + \frac{\partial F^{32}}{\partial x^2} + \frac{\partial F^{33}}{\partial x^3} \\
&= -\frac{1}{c^2}\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} + 0 \\
&= \mu_0 J_z
\end{aligned}
$$

These are the results when we take the derivatives with respect to the second index on $F^{\mu\nu}$, differentiating across the rows. What if we differentiate across the columns with $F^{\mu\nu}{}_{,\mu}$? This is a simple matter of applying the antisymmetry: $F^{\mu\nu}{}_{,\mu} = -F^{\nu\mu}{}_{,\mu}$. Upon careful inspection, we find that

$$
F^{\nu\mu}{}_{,\mu} = \mu_0 J^\nu = \mu_0 \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}
$$

or, the derivative across one index of the electromagnetic field strength tensor provides the relativistic current density four-vector. The time component of this vector is directly related to the charge density producing the electric field. Tracking the subscripts while taking another derivative, we find that

$$F^{\mu\nu}{}_{,\nu,\mu} = \mu_0 J^{\mu}{}_{,\mu} = 0 \tag{5}$$

which is equivalent to equation 1 on page 4 and equation 4 on page 4 combined.

A similar equation exists for the dual tensor.