Cosmology
Version A: Full Math

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April 21, 2012

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1 What Is Cosmology?

Cosmology is the study of the structure and nature of the universe. Instead of focusing on any single object, cosmologists study the Universe as a whole and the way it evolves over time.

2 Einstein's Cosmological Constant

With the discovery of the general theory of relativity, the geometry of the universe was quantifiable and definable. This meant that, for the first time, the structure of the Universe could be studied explicitly. Einstein was amongst the first to do this calculation, and he ran into a result that disturbed him greatly. When he examined the geometry of the Universe we live in, he uncovered math that made it appear as though the Universe was expanding. This did not fit his ideas of theology: Einstein felt that the God he believed in would have created a perfect Universe to begin with, and expansion represented an undesirable form of change. The equations included a constant whose value could not have been measured at the time. Einstein arbitrarily chose a negative value for this constant which would result in a static Universe which would prevent this predicted expansion.

A few short years later, Edwin Hubble published his results which proved that the Universe was, in fact, expanding. This shocked Einstein and others. The Universe was an evolving beast of some form. Einstein and others recognized his earlier error and made another one: they arbitrarily decided that the cosmological constant should have a value of zero. This led to an expanding Universe model, and everything seemed fine. It seemed clear that, at some point in the past, the Universe had been a finite point, which then exploded in a Big Bang, and began expanding from that point.

This was the accepted model for almost 80 years. It was challenged in 1998, when a group of scientists recognized that the value of zero is just as arbitrary and any other value for the cosmological constant, so they set about trying to measure the value explicitly. What they discovered shocked just about everybody: the cosmological constant is not zero. Not only that, it is a positive number. In other words, the Universe would expand whether it contained matter or not. The question is: why? What physical process or object does this cosmological constant represent?

This is still one of the unanswered questions of cosmology. There have been a number of proposals, most of which involve dark matter or dark energy. Astronomical observations show expansion behaviour that cannot be accounted for by the matter and energy we can observe. Therefore, it has been proposed that there is some sort of matter or energy out there which we cannot observe, which is commonly referred to as dark matter and/or dark energy. Very little is known about dark matter, although much has been suggested. The difficulty with studying dark matter is that it is dark; it cannot be directly observed or studied using today's technology. This may change in the future as new technologies are developed, but that day still appears to be a long way off.

2.1 Expanding Into What?

One of the philosophical questions that has been asked is what the Universe is expanding into. Is there a larger universe with greater dimensions outside of this one? If so, what is that Universe "in?" Is there an infinite chain of universes, like some infinite set of matryoshka dolls?

The question about what is outside our Universe, if anything, will probably never be answered. Our instruments are limited to measuring that which is in our Universe, and that alone. It has been shown, however, that it is possible for the complete "exterior" geometry of such a Universe to remain unchanged, while the "internal" geometry continues to expand. If there is an outside Universe, our presence in that Universe may not appear to be changing at all.

3 The End(s) of Time

One of the most commonly asked questions in cosmology is "how will the Universe end?" There have been a number of proposals for this.

- **The Big Crunch** - In this theory, the expansion of the Universe eventually slows, and the Universe starts to fall back in upon itself, destroying all life in the process.[1] This theory has a companion theory, in which the lifetime of our Universe is cyclic, and the Big Crunch would be followed by a new Big Bang, with a new Universe being formed. This would happen over and over again in an infinite cycle.
- **The Static Universe** - This theory proposes that there is just enough mass around to slow the expansion of the Universe without ever quite halting it. This, ultimately, leads to a Universe very much like the static Universe Einstein's theological instincts wanted. It wouldn't technically be static, but the expansion would eventually become imperceptibly slow.

[1] Unless you are a being named Galan who likes to snack on planets. Then you might survive.

In this case, thermodynamics predict that all energy would eventually be converted into unrecoverable waste heat due to entropy, wiping out all life in the "heat death" that results from all of existence boiling away into a plasma that no longer allows complex molecular structures to exist.

- **The Constantly Expanding Universe -** In this theory, the Universe simply continues to expand, causing an increasing space between civilizations and worlds, and reducing the average energy in any given volume. Life, in this model, would eventually die off when all temperatures in the Universe drop below the point at which chemical reactions can occur and all of existence is frozen out at temperatures very near absolute zero.

The discovery of a positive cosmological constant make the third theory the most likely theory. Tell your descendants to bundle up: it's going to get cold.

4 The Shape of the Universe

The simplest cosmology to work with in general relativity is a combination of the work of Einstein, Robertson and Walker. Robertson and Walker added two assumptions to Einstein's field equations:

1. At any instant in time, the entirety of space appears homogeneous and isotropic, barring the effects of mass an energy within it. In other words, any two locations in an empty universe are utterly indistinguishable from each other.
2. Our definition of simultaneity makes sense from the perspective of this universe.

These assumptions ultimately result in a metric that gives the line element

$$dl^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2$$

where $d\Omega$ is the unitless volume element of the spatial components of the spacetime, and k is defined by

$$g_{rr} = \frac{1}{1 - kr^2}$$

as a representation of the spatial dependence of the curvature of spacetime, relative to the "centre" of this cosmology.

The assumptions provided do not allow us to specify a value for k alone. It is here that the different possibilities for the future of the Universe appear; the value of k determines the shape of the cosmology.

In the case in which $k = 1$, the spatial extent of the universe at a given point in time is a perfect sphere. This is a closed spacetime; the universe folds back in on itself, indicating that a sufficiently fast and long-lived individual can travel along a geodesic, circumnavigate the universe, and return to his or her starting point, just as a person can walk (and/or swim) in the straightest line possible starting at Earth's North Pole and return to that North Pole later. Other positive values of k also result in closed universes, but they are not perfectly spherical. At the point $k = 0$, the universe is flat Minkowski space as in the world of special relativity. Finally, if $k = -1$, the universe is a perfect hyperbolic shape, never ending or closing in upon itself in what is called an "open" geometry.

There are certainly other cosmologies, but most of them involve assumptions which are less intuitive. That doesn't mean they are incorrect, but it does mean further evidence will be required before one can find a good candidate amongst them to describe our universe.

5 The Fate of the Universe

The k parameter described above is one of the two parameters that are significant when it comes to describing the ultimate fate of the Universe. The second is Einstein's famous, or possibly infamous, cosmological constant.

To see how the cosmological constant arises, we need the stress-energy tensor. We can treat spacetime as a fluid, which is pulled, pushed and twisted around by gravity. This can be described by the stress energy tensor $T^{\alpha\beta}$ which describes the flux of momentum p^α over a surface of constant x^β.[2] This definition and the Ricci tensors are enough to show that the ultimate gravitational fields $G^{\alpha\beta}$ are related to the metric and stress energy tensor by

$$G^{\alpha\beta} = kT^{\alpha\beta} - \Lambda g^{\alpha\beta}$$

where Λ is a constant of integration that appears along the way. Einstein set this constant, now known as Einstein's cosmological constant, to a specific value that created a "steady state" universe, meaning that the Universe would not expand. After Hubble's discovery, the scientific community started treating the equations as though $\Lambda = 0$, but that choice is just as arbitrary. In fact, recent attempts to explicitly study and measure this constant indicate that it is not only non-zero, but that it has a positive value sufficient to force the expansion of an empty universe.

Nonetheless, it is the combined values of k and Λ that ultimate determine the fate of the universe as it expands or contracts for the rest of time. At this

[2] See any fluid dynamics course for complete details of how to define this.

time, we do not have sufficient empirical evidence to determine which possibility is the real one, or even of the Robertson-Walker cosmology is the correct one; some of those exotic assumptions leading to other cosmologies may turn out to be correct.