The Postulate of General Relativity
Version A: Full Math

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Contents

1 Terminology: Special and General

What makes "special relativity" special? What makes "general relativity" general? These are common questions when one first researches the subjects, particularly when one is coming from a non-mathematical background.

In mathematical situations, research is often devoted to special and general cases. A "special" case is one in which specific values are known, or specific restraints are applied. For example, most elementary school math deals with special cases. Students solve questions such as 234×123 instead of solving the general case of $a \times b = c$ and then substituting $a = 234$ and $b = 123$ before solving for c.[1] Science works in a similar way. Special relativity considers the special case in which the reference frames one uses to measure from are experiencing uniform, or unaccelerated, motion. General relativity expands the theory to allow observers to move with arbitrary accelerations.

2 The Postulate of General Relativity

The distinction between special and general relativity boils down to a single realization: one cannot distinguish between different accelerations by feel alone. Imagine being placed in a sealed and soundproof room (with ample oxygen supply.) Imagine also that you can distinguish between "up" and "down" as directions. There is a particular side of this room that you can call a "floor." Einstein realized[2] that there is no physical sensation that can be used to distinguish between standing in a sealed room on the surface of a planet and standing in a sealed room with a rocket engine attached that forces the room to accelerate.[3] This equivalence led to the formulation of *non-inertial* reference frames, or accelerated reference frames.

[1] It is the author's opinion that the lack of explicit instruction about the usefulness and need for general cases rather than exclusively special cases is one of the main reasons students struggle with algebra. When faced with a massive paradigm shift in mathematical thinking that is not *treated* as a massive paradigm shift for fear of scaring students away, the students become convinced that algebra is an exercise in futility and harder than it needs to be. When the author taught algebra to first time learners, he started with lessons about the paradigm shift and the axioms of algebra and the need for special cases before getting into the nitty gritty steps. He then gave out a homework assignment asking students to manipulate high school physics formulae in the general case only, and the class average was 87%. When he later taught high school physics to a completely different group of students and gave those students the identical assignment, the class average was 63%. End rant.

[2] He was riding an elevator at the time.

[3] We are forced to assume that any noises or vibrations caused by the rocket engine are not transmitted to the room.

3 Gravity

Beyond allowing the formulation of non-inertial reference frames, this postulate connected gravity to reference frames, which were already connected to the speed of light. This was a monumental connection. At the time, only three forces were recognized in science: there was the electrical force, the magnetic force and the gravitational force. With the first two bound by relativity, scientists began searching for a unified theory that could bind the “final” force to the other two.[4] Developing this “Grand Unified Theory,” or GUT, became Einstein’s unrealized goal in life. Over 50 years after his death, such a theory has still not been formulated to the satisfaction of most physicists.

This formulation did, however, spawn a new revelation: mass and energy do not merely exist “in” the geometry of reality, they alter and define the geometry of reality. Imagine you are still in your earliest school years, and have been tasked with drawing a triangle. The implications of Einstein’s postulate are akin to watching your piece of paper change and transform in shape and size as you draw the triangle on its surface. A flat piece of paper ceases to be flat as soon as something is drawn on it, instead curving in a way that attracts all other shapes on the page to the one you have just drawn. It is this revelation that has led to the notion of the “fabric” of spacetime. The mental image conjured by the mathematics renders reality as a great sheet. Massive and/or energetic objects dent this sheet, altering the “fabric” of the sheet of reality.

4 The Geometry of Reality

This, once again, changed the way the geometry of the world was laid out. “Rotating” objects to different speeds caused ripples in the sheet, moving and shifting the dents in what we perceive as gravity. This even led to the theory of gravitational waves, which emanate from a source of gravity and propagate throughout the universe.

These dents and ripples have a great impact on the motion of objects. It is no longer natural to think of objects thinking in straight lines. They can move in the straightest possible lines, but that isn’t quite the same thing; one can drive a “straight” stretch of road for several kilometers, but odds are the road curves along the surface of the Earth as elevations change on the oblate spheroid

[4]By today’s count, there are four forces: the gravitational force, the electromagnetic force, the weak nuclear force and the strong nuclear force. Because electricity and magnetism are already connected and unified in our day to day situations, they are now treated as a single force. They have been unified with the weak and strong nuclear forces as well, but this unification is only visible under the extreme conditions created within particle accelerators, and so they are still treated as distinct forces in most circumstances.

we call home. We now talk about "geodesics," which are the straightest possible lines one can follow on a curved surface.

To thoroughly drive the strangeness of this concept home, we will look at a shape that cannot exist on a flat surface, but which every reader is likely familiar with without even realizing that this is the case. The shape we are talking about is a "biangle." This shape is formed when two lines are drawn as straight as possible (i.e. two geodesics) and meet in exactly two places. In the geometry taught in public school, this cannot happen, but the geometry taught in public school is restricted to the "special case" of flat surfaces. The "general case" of geometry allows this shape to exist. You will probably not find examples in your mathematics classroom, but are extremely likely to find them in your geography classroom.

Take a globe of the Earth. It is covered with lines of longitude and latitude. Some, but not all, of these lines are geodesics. We need to determine which are which. Imagine you have a globe from the factory before it has been painted or marked in any way. It is a perfect sphere. You take a marker and start drawing the straightest possible line that you can from any point on the surface. You will eventually return to the spot you started from.[5] As the globe started with no distinguishing features, all such geodesic lines would be identical when drawn carefully and accurately. This is the case with lines of longitude on a globe, and *not* most[6] lines of latitude. If one takes two lines of longitude for examples on the globe, you will find that they form a biangle; they meet at the north pole, continue along the straightest possible paths until they are parallel while crossing the equator, and then converge to meet at the south pole and complete the biangle.

Similarly, "rules" for shapes that we are aware of do not necessarily apply. For example, we are taught that all triangles have angles which add up to 180°. While this is absolutely true on a flat surface, one can easily form a triangle with three right angles on an appropriately curved surface. Begin at any point on the equator. Travel directly to the north (or south) pole. Make a 90° turn and continue until you reach the equator once more. Make another 90° turn (in either direction, East or West) and return to your starting point. The triangle mapped out by these three geodesics will have three right angles, for a total angle of 270°.

We can take this same issue one step further and demonstrate the "parallel transport problem." Again, imagine you are standing on the equator, facing North with your right arm held out in the Eastward direction. Walk to the north pole, and with each step, you keep your arm held out parallel to the

[5] Note that the author assumes your penmanship and artistic skills are significantly better than his own. He'd miss the target by a significant amount.

[6] There is a grand total of one line of latitude that forms a geodesic. Deducing which line that is indicates a good understanding of the topic.

direction is was in when you took your previous step. Upon reaching the north pole, turn right, keeping your arm parallel to its original direction so that it now points directly ahead of you. Walk until you reach the equator and turn right again; you will now either have your right arm across the front of your body, or you will extend your left arm to maintain the same direction.[7] Walk back to your starting point. Your extended arm has been held parallel to its previous direction every step of the way, and yet the arm that started out pointing East is now pointing South! This is the parallel transport problem: you cannot transport something pointed a particular direction and guarantee that its direction will remain constant across a curved surface. Those paths which keep their own direction constant (i.e. those that don't "feel" like turning to a walking person: if you point your arm straight ahead with your first step, it points straight ahead of you at every step; if you return to your original position, you are facing in your original direction when you do) are the geodesics for that surface.

The concept of the geodesic is not just one that seems correct, but is one that turns out to be fundamental to the behavior of objects in our universe. If we go back to Newton's original axioms, he stated "Unless an outside force is applied, a stationary object (object "at rest") will remain stationary and a moving object will continue moving at the same speed and in the same direction." In short, things move in straight lines. We now modify this to say that, in the absence of outside forces, objects move along geodesics.

This counter intuitive geometry had been explored by abstract mathematicians for centuries before its application to reality through relativity. With this toolkit available, physicists soon discovered theoretical objects that still hold imaginations enthralled to this day: black holes and worm holes.

5 Our New Coordinate Axes

The implications of Einstein's postulate of general relativity impact the geometry of space-time. Those implications are reflected in our mathematical constructs through the metric tensor. To this point, our metric has been defined as

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

None of the entries in this metric are variable in any way. This implies that our Universe is completely uniform and unchanging, and that space is not

[7] Your right arm is probably pretty sore at this point.

affected by the arrangement of objects within it. We now realize this is not the case. Thus, we need to construct a new metric that represents real life situations. This brings us even further from Euclidean space.

To this point, our reference frames have been based on four axes: ct, x, y and z. These will no longer be the most natural axes. Space is distorted by gravity, and the most easily studied sources of gravity are stars, planets, moons, and similar astrophysical objects. These tend to share one thing in common: their self-gravitational force is powerful enough to ensure that they are reshaped into approximately spherical objects. Thus, the spherical coordinate system is the one that seems appropriate to most situations.[8]

In the spherical coordinate system, our imaginary axis ct will remain unchanged. We will divide the spatial coordinates into three new directions, which are most easily imagined using a globe of the Earth. Pick a point anywhere on the Earth's surface. The three coordinates are described by numbers r, θ and ϕ. The r coordinate is the radial axis, describing the distance from the centre of the Earth to the point chosen. In this case, the value of r would be the radius of the Earth $r_E = 6.38 \times 10^6$ m. The direction is along the line from the centre of the Earth to the selected point, which means the direction is variable. We've been dealing with coordinate systems that change all along, but to this point, they have only changed as they move with respect to each other, and not with respect to points measured by the coordinate system. This is no longer an option in general relativity; our coordinate systems must have the freedom to move around. It is customary to keep r as a non-negative number. It may be either zero or positive; negative values are represented by positive r values with different values for the angular coordinates we will now define.

The second coordinate, θ, is an angle describing the elevation relative to the line connecting the north and south poles. On a globe of the Earth, this is represented by the line of latitude involved. Specifically, it is measured relative to the line connecting the centre of the Earth with the north pole, and it is always taken to be a non-negative number. Thus, the tropic of cancer ($23.45°$ north latitude) would be at $66.55°$ if this angle were to be measured in degrees. The equator would be at $90°$, and the tropic of capricorn would be at $113.45°$. The south pole would be at $180°$, the largest possible value. The direction of this axis is parallel to Earth's surface, pointing south, the direction of increasing ϕ.

The third coordinate, ϕ, is an angle that measures rotation with respect to some arbitrary point around the surface, represented on Earth by lines of longitude. Angles are measured with respect to the Greenwich Mean Line, and rotate around the Earth from there. While Earth's geographers chose a

[8] As we shall see, the best coordinate system is one created specifically for this application. It grows out of the spherical coordinate system, so that is the one we shall use.

standard that works with positive and negative values (marked "East" and "West"), mathematicians and physicists prefer to keep all values positive. If one looks down on the Earth from above the North Pole, then the coordinate ϕ increases as you move counter clockwise around the globe. The allowed values of ϕ would range from zero to 360°, representing a full circle, if they were measured in degrees at all; more on that later. The direction of this axis is parallel to the line of latitude on the surface of the Earth in the direction of increasing ϕ, which is East on the surface of the Earth.

The metric describing this system is one which must be derived. Before this is possible, we must work through a number of intermediate steps.

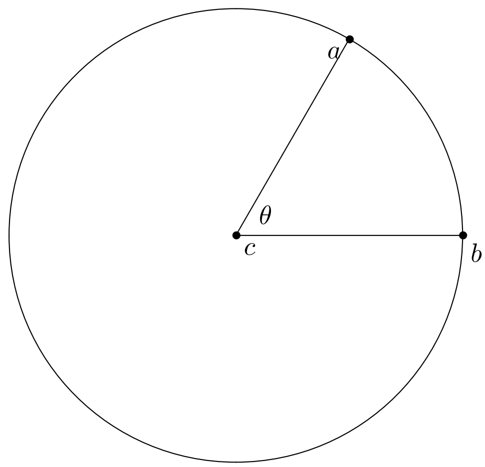
6 Radian Angular Measure

Angles are frequently measured in degrees in our day to day experiences, ranging from 0° to 360°. This is an ancient and convenient convention, chosen arbitrary by the humans who developed it.[9] This is also a system that has a deep but subtle flaw. When performing algebra with angles, the variable cannot appear outside a trigonometric function. In normal day to day applications, such as the whole of engineering, instances in which angles should appear outside trigonometric functions are extraordinarily rare, but in general relativity and theoretical mathematics, they occur regularly enough that we need a more natural means to measure angles. This natural angle is the radian.

The circle is a well known geometric shape. Every circle comes in the same proportions: the circumference around the outside C is directly proportional to the radius r through the relationship $C = 2\pi r$. This relationship forms the basis of the most natural angular measure. Imagine a circle with points a and

[9] It is frequently taught that the 360 point limit was chosen to approximate the angle Earth swept through its orbit in one day. While this sounds reasonable, it does not appear to be the case; the earliest records of the 360 degree system are found in cultures that did not use the 365 day calendar. The cultures lacked fractions, so it appears 360 was chosen because a single degree is large enough to measure, but the circle can still be divided by a large number of divisors, as 360 is divisible by 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18 and 20.

b on the circumference:



The length of the segment of the circumference s between points a and b is determined by two variables: the radius r, and the angle θ. Thus, radians are defined by the relationship between this segment's length and the radius of the circle:

$$\theta = \frac{s}{r}$$

Thus, the total angle in the centre of a circle is the angle that one gets when the segment is the full circumference of the circle:

$$\theta = \frac{2\pi r}{r} = 2\pi$$

Thus,

$$2\pi \text{ rad} = 360^\circ$$

or

$$1 = \frac{360^\circ}{2\pi \text{ rad}} = \frac{180^\circ}{\pi \text{ rad}}$$

which is the conversion factor that is familiar to those who have seen radian measures before.

7 General Lorentz Transformation Tensors

The Lorentz transformation tensor we've seen previously

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

can be generalized into a tensor that converts between *any* two coordinate systems. We can convert from the system with basis vectors $\mathbf{e}_\nu$ to the system with basis vectors $\mathbf{e}_{\mu'}$ by defining each component of the Lorentz transformation tensor as

$$\Lambda^{\mu'}{}_{\nu} = \frac{\partial \mu'}{\partial \nu}$$

for each possible value of μ' and ν. To see how this is applicable, we will calculate the conversion factors from three dimensional Cartesian coordinates (ν) to three dimensional spherical coordinates (μ'). This requires first defining r, ϕ and θ in terms of x, y and z, and vice versa.

It is easiest to define the Cartesian coordinates in terms of the spherical coordinates:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

We can manipulate these equations to find the spherical coordinates in relation to the Cartesian coordinates:

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

We can now define the general Lorentz transformation tensors that connect our two coordinate systems. We start with the tensor that converts cartesian coordinates to spherical coordinates. As basis vectors have lowered indices, this would be the transformation

$$\Lambda^{\mu}{}_{\nu'}\mathbf{e}_\mu = \mathbf{e}_{\nu'}$$

where primed indices correspond to the spherical coordinate system. Thus,

$$\begin{aligned} \Lambda^{\mu}{}_{\nu'} &= \begin{bmatrix} \Lambda^{x}{}_{r} & \Lambda^{x}{}_{\theta} & \Lambda^{x}{}_{\phi} \\ \Lambda^{y}{}_{r} & \Lambda^{y}{}_{\theta} & \Lambda^{y}{}_{\phi} \\ \Lambda^{z}{}_{r} & \Lambda^{z}{}_{\theta} & \Lambda^{z}{}_{\phi} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} \\ &= \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} \end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{e}_r &= \Lambda^{\mu}{}_{r}\mathbf{e}_{\mu} \\
&= \frac{\partial x}{\partial r}\mathbf{e}_x + \frac{\partial y}{\partial r}\mathbf{e}_y + \frac{\partial z}{\partial r}\mathbf{e}_z \\
&= \sin\theta\cos\phi\mathbf{e}_x + \sin\theta\sin\phi\mathbf{e}_y + \cos\theta\mathbf{e}_z
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{e}_\theta &= r\cos\theta\cos\phi\mathbf{e}_x + r\cos\theta\sin\phi\mathbf{e}_y - r\sin\theta\mathbf{e}_z \\
\mathbf{e}_\phi &= -r\sin\theta\sin\phi\mathbf{e}_x + r\sin\theta\cos\phi\mathbf{e}_y
\end{aligned}$$

We can construct the metric of spherical coordinates knowing that

$$g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$$

and find

$$\begin{aligned}
g_{\mu\nu} &= \begin{bmatrix} \mathbf{e}_r\cdot\mathbf{e}_r & \mathbf{e}_r\cdot\mathbf{e}_\theta & \mathbf{e}_r\cdot\mathbf{e}_\phi \\ \mathbf{e}_\theta\cdot\mathbf{e}_r & \mathbf{e}_\theta\cdot\mathbf{e}_\nu & \mathbf{e}_\theta\cdot\mathbf{e}_\phi \\ \mathbf{e}_\phi\cdot\mathbf{e}_r & \mathbf{e}_\phi\cdot\mathbf{e}_\theta & \mathbf{e}_\phi\cdot\mathbf{e}_\phi \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}
\end{aligned}$$

The reverse transformation is facilitated by

$$\begin{aligned}
\Lambda^{\mu'}{}_{\nu} &= \begin{bmatrix} \Lambda^{r}{}_{x} & \Lambda^{r}{}_{y} & \Lambda^{r}{}_{z} \\ \Lambda^{\theta}{}_{x} & \Lambda^{\theta}{}_{y} & \Lambda^{\theta}{}_{z} \\ \Lambda^{\phi}{}_{x} & \Lambda^{\phi}{}_{y} & \Lambda^{\phi}{}_{z} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{bmatrix} \\
&= \left[\begin{matrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} \\ \frac{xz}{(x^2+y^2+z^2)^{\frac{3}{2}}\sin\left(\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right)} & \frac{yz}{(x^2+y^2+z^2)^{\frac{3}{2}}\sin\left(\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right)} \\ -\frac{y}{x^2\sec^2\left(\tan^{-1}\left(\frac{y}{x}\right)\right)} & \frac{1}{x\sec^2\left(\tan^{-1}\left(\frac{y}{x}\right)\right)} \end{matrix} \right. \cdots \\
&\qquad \cdots \left. \begin{matrix} \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \frac{z\left(x-\sqrt{x^2+y^2+z^2}\right)}{(x^2+y^2+z^2)^{\frac{3}{2}}\sin\left(\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right)} \\ 0 \end{matrix} \right]
\end{aligned}$$

which is far simpler to calculate than it would appear.

We are now equipped to convert between the two coordinate systems. This is not sufficient for work in relativity, though. We often wish to work in two coordinate systems simultaneously, such as those in our S and S' frames. This is the framework that we still need to establish.

8 Christoffel Symbols

Let us examine

$$\frac{\partial \mathbf{e}_x}{\partial \theta}$$

in detail. As $\mathbf{e}_x$ is constant, the derivative should come to zero regardless of coordinate system.[10]

Using our above expressions written in spherical coordinates, we find that

$$\begin{aligned}
\mathbf{e}_x &= \frac{\partial r}{\partial x}\mathbf{e}_r + \frac{\partial \theta}{\partial x}\mathbf{e}_\theta + \frac{\partial \phi}{\partial x}\mathbf{e}_\phi \\
&= \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{e}_r + \frac{xz}{(x^2+y^2+z^2)^{\frac{3}{2}}\sin\left(\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right)}\mathbf{e}_\theta \\
&\quad - \frac{y}{x^2\sec^2\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}\mathbf{e}_\phi \\
&= \sin\theta\cos\phi\mathbf{e}_r + \frac{\cos\theta\cos\phi}{r}\mathbf{e}_\theta - \frac{\sin\phi}{r\sin\theta}\mathbf{e}_\phi
\end{aligned}$$

One might then assume

$$\frac{\partial \mathbf{e}_x}{\partial \theta} = \cos\theta\cos\phi\mathbf{e}_r - \frac{\sin\theta\cos\phi}{r}\mathbf{e}_\theta + \frac{\cos\theta\sin\phi}{r\sin^2\theta}\mathbf{e}_\phi$$

but one would be wrong. The problem is that the basis vectors are *not* constant. Thus, the proper derivative is

$$\begin{aligned}
\frac{\partial \mathbf{e}_x}{\partial \theta} &= \cos\theta\cos\phi\mathbf{e}_r - \frac{\sin\theta\cos\phi}{r}\mathbf{e}_\theta + \frac{\cos\theta\sin\phi}{r\sin^2\theta}\mathbf{e}_\phi \\
&\quad + \sin\theta\cos\phi\frac{\partial \mathbf{e}_r}{\partial \theta} + \frac{\cos\theta\cos\phi}{r}\frac{\partial \mathbf{e}_\theta}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial \mathbf{e}_\phi}{\partial \theta}
\end{aligned}$$

[10] Remember, the spherical and Cartesian coordinates do not *move* with respect to one another. The spherical unit vectors are not constant, but they don't move per se.

It is this complete combination that is identically zero, as

$$\begin{aligned}
\frac{\partial \mathbf{e}_r}{\partial \theta} &= \frac{1}{r}\mathbf{e}_\theta \\
\frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -r\mathbf{e}_r \\
\frac{\partial \mathbf{e}_\phi}{\partial \theta} &= \frac{\cos\theta}{\sin\theta}\mathbf{e}_\phi
\end{aligned}$$

The final details of the proof left to the reader. It is a matter of a large number of trigonometric gymnastics.

The most important point to take away from the above is that systems with variable basis vectors have more involved vector and tensor derivatives than other systems. Thus, we introduce the Christoffel symbols to facilitate such derivatives:

$$\Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} = \frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} \tag{1}$$

Just as we defined

$$A^{\mu}{}_{,\nu} = \frac{\partial A^{\mu}}{\partial x^{\nu}}$$

in Cartesian space with its constant basis vectors, we can now define

$$A^{\mu}{}_{;\nu} = A^{\mu}{}_{,\nu} + A^{\alpha}\Gamma^{\mu}{}_{\alpha\beta}$$

as the most useful total derivative in systems with variable basis vectors.

It should be noted that this definition of Christoffel symbols can often be extremely cumbersome. At times, it is easier to computer the Christoffel symbols using the identity

$$\Gamma^{\gamma}{}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right) \tag{2}$$

Similarly, the general divergence of a vector can be defined as

$$\vec{\Box}\cdot\vec{A} = \Box_{\mu}A^{\mu} = A^{\mu}{}_{;\mu} = A^{\mu}_{,\mu} + \Gamma^{\alpha}{}_{\mu\alpha}A^{\mu}$$

9 Pseudo-Riemannian Manifolds

We can now start to examine manifolds. A "manifold" in mathematics is a surface within a higher dimensional space. If you take a sphere of a given radius, then the surface of that sphere is a two dimensional manifold: the r coordinate can be assumed, so the position on the surface can be described completely by the two coordinates θ and ϕ. A sufficiently small observer standing on the surface (such as humans on the surface of the Earth) could easily mistake this surface for a flat one. This last property, in which "local" coordinates appear to fit Minkowski metrics, make the surface pseudo-Riemannian. A Riemannian manifold is one in which every vector dot product (i.e. $A^\mu A_\mu$) is positive definite for non-zero vectors; this property never applies in relativity, as light-like vectors have 0 dot products and time-like vectors have negative dot products (in our chosen convention.) The full derivation and definition of such objects will be quite abbreviated in this treatment, omitting most proofs and derivations for the sake of brevity.[11] The basic mechanism of the proof is similar to working out the first derivative in calculus from first principles; set up a surface, look at points (x, y), $(x + \delta x, y)$, $(x, y + \delta y)$ and $(x + \delta x, y + \delta y)$, and then work out the vector differences between the two points and the surface vectors that describe them.

It can be shown that a vector V^α is kept parallel to its original direction as it is carried along a curve U^β if the equation

$$U^\beta V^\alpha{}_{;\beta} = 0$$

holds along the entire curve. The curve U^β is a geodesic if it can transport its tangent vector in this fashion, such that

$$U^\beta U^\alpha{}_{;\beta} = U^\beta U^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\mu\beta} U^\mu U^\beta = 0$$

This places constraints on U^β that depend upon the Christoffel symbols. The Christoffel symbols, in turn, depend on the metric, and the metric defines the curvature of the space-time or manifold.

Sufficient algebraic gymnastics can be performed to define the curvature of the space in terms of the *Riemannian tensor*. This tensor is most naturally defined in terms of the Christoffel symbols, but can also be defined in terms of the metric, as follows:

$$\begin{aligned} R^\alpha{}_{\beta\mu\nu} &= \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\sigma\mu}\Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\beta\mu} \\ R^\alpha{}_{\beta\mu\nu} &= \frac{1}{2} g^{\alpha\sigma} \left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\nu} \right) \end{aligned}$$

This Riemannian tensor is constructed of combinations of the symmetric metric tensor which differ by signs. That means it has a number of symmetries and antisymmetries of its own. These can be summarized with a couple of

[11] Readers looking for more complete details are encouraged to track down *A First Course in General Relativity* by Bernard F. Schutz, ISBN 0-521-27703-5.

identities, once one has established that $R_{\alpha\beta\mu\nu} = g_{\alpha\sigma} R^{\sigma}{}_{\beta\mu\nu}$ is a valid way to lower the index on $R^{\alpha}{}_{\beta\mu\nu}$:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$$
$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

The specific entries in this tensor are the ones that describe the curvature in space. In space that isn't curved, the Christoffel symbols are all 0, and so is the Riemann tensor. Every straight line is a geodesic and parallel lines are preserved.

10 The Metric of Newtonian Gravity

We are finally equipped to create the equations that describe curvature in space that has been generated by Newtonian gravity.[12] We know that Newton's gravitational equations work well for low speeds and relatively weak gravitational sources, as they have worked very well for years. We also know, thanks to Mercury's orbit, that they are not a complete description. In our first step, we construct a metric which gives us Newtonian gravity to first order, and then we will generalize to other cases later. One of the assumptions that we are required to make when constructing the metric is that the gravitational field does not change with time. This will turn out to be the vital difference between Newtonian and relativistic gravity.

Generally speaking, gravity is a weak force. We construct our approximation theory of gravity by adding terms related to the strength of the field f to the Minkowski metric as follows:

$$g_{\mu\nu} = \begin{pmatrix} -1 - O(f) - O\left(f^2\right) - \dots & 0 & 0 & 0 \\ 0 & 1 + O(f) + O\left(f^2\right) + \dots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $O(f^n)$ is read "terms of order f^n" and means "terms that include a factor of f^n." For Newtonian gravity, we will only concern ourselves with terms of order f, and in which all higher powers are negligible in our day to day lives.[13] The field strength should remain unchanged when compared to its Newtonian form, so

$$f = -\frac{GM}{r}$$

[12] It truly will be more "creation" than derivation. A full derivation would require either more than nine lessons in this series, or absolutely gargantuan lessons.

[13] This is not an unjustified assumption: this is exactly what we already have when comparing Newtonian equations for kinetic energy, velocity, acceleration, etc. to their relativistic counterparts.

where $G = 6.67 \times 10^{-11}$ N m^2/kg^2 is Newton's gravitational constant, M is the total *inertia* of the object rather than its mass, and r is the distance from that source of mass/inertia M. If we are adding this term to the dimensionless constant 1, or subtracting it from -1, we need to apply a coefficient to render this dimensionless. The overall units on f are

$$\text{N m/kg} = \text{kg m}^2 \ / \ \text{s}^2 \ \text{kg} = \text{m}^2/\text{s}^2$$

which is a strong indication of the constant to use. With the addition of a factor of 2, justified solely by testing this result and finding that the factor is required to conform to Newtonian gravitational results, we arrive at the final form of our metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{2GM}{c^2 r} & 0 & 0 & 0 \\ 0 & 1 - \frac{2GM}{c^2 r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If we place our source of gravity at the origin of our coordinate system, then the metric simplifies to

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{2GM}{c^2\sqrt{x^2+y^2+z^2}} & 0 & 0 & 0 \\ 0 & 1 - \frac{2GM}{c^2\sqrt{x^2+y^2+z^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

To see how this describes the action of bodies, we recall our modified axiom: objects which experience no outside forces move along geodesic curves, conforming to the geodesic equation 9 on page 13. Imagine a body at rest with position vector

$$\vec{x} = \begin{pmatrix} 0 \\ x \\ 0 \\ 0 \end{pmatrix}$$

We now track the behavior of this object over time through the geodesic equation. We force ${x^\alpha}_{;\beta} = 0$ to be true, which leaves the constraint

$${x^\alpha}_{,\beta} + {\Gamma^\alpha}_{\mu\beta} x^\mu x^\beta = 0$$

or (given that x^1 is the only component of x^μ which is not zero)

$$\frac{d}{d\tau} x = -{\Gamma^1}_{11} x^1 x^1$$

By equation 2 on page 12,

$${\Gamma^1}_{11} = \frac{1}{2} g^{11} \left(g_{11,1} + g_{11,1} - g_{11,1} \right) = \frac{1}{2} g^{11} g_{11,1}$$

which results in the expression

$$\frac{dx}{d\tau} = -\frac{GMc}{c^2r - 2GM}$$

which implies exactly what we wanted: an object sitting freely in the vacuum will move to the origin of the coordinate system whether a force acts upon it or not. The shape of the space itself demands movement toward the origin. Gravity is now built into the shape of the universe at its most fundamental level.

This is probably the last time we will be using Cartesian coordinates as our main coordinate system. As soon as the gravitational field changes over time, some of our assumptions break down. Furthermore those broken assumptions mean we can no longer ignore the higher order terms (which, for terms of order f^n, includes terms of order c^{-2n}, which is why we don't notice them) as speeds really start to approach those of light, and our dimensionless higher order terms start to take on values that are closer and closer to 1. These effects will be examined closely in the context of black holes and worm holes.