Holes in Space
Version A: Full Math

W. Blaine Dowler

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1 Black Holes

Scientists and science fiction writers have been fascinated by black holes since before they even had names. Some of these writers did insufficient research, and as a result, popular understanding of black holes includes a number of misconceptions. Before we begin to examine the reality and the fiction surrounding these objects, however, we must first establish some quantum mechanics.

1.1 Quantum Mechanics

All objects with mass are made up of a collection of subatomic particles, the most common of which are electrons, protons and neutrons. It is less common knowledge that protons and neutrons are, themselves, made up of subatomic particles known as quarks, and that the right conditions can force a proton and electron to combine to form a neutron. The universe is also filled with "virtual particles," which appear temporarily, generally in pairs, and allow objects to feel forces, among other things.

As surprising as it may be, these incredibly tiny objects serve important roles in the study of black holes. We will deal with virtual particles in section 1.4 on page 5. For now, we will only examine the effects of great pressure on atoms and nuclei.

Stars are the most massive common objects in our galaxies. They release energy due to natural fusion reactions. In short, they have so much mass that the atoms in their cores are crushed together into larger particles. The heat energy this releases helps the star maintain its size and structure. As the atoms within the star get larger and larger, it requires more and more energy to force them to combine into larger atoms through fusion. Stars eventually "run out of fuel," meaning that the atoms in their core are too large to be forced to fuse together by the star's gravity any longer. The star "dies" as the internal pressure suddenly drops, setting off a chain reaction that leads to a nova or supernova explosion as the internal pressure suddenly drops. What remains is a massive husk, rife with gravitational pressures, but lacking the internal fusion that provides a counter pressure to keep the star at its original size. If this husk is large enough, it forms a neutron star; all of its internal atoms are gravitationally crushed together until the core is a single giant nucleus made solely out of neutrons. In some cases, the husk has too much mass to even exist as a neutron star, and it continues to collapse. These objects become black holes.

1.2 What Black Holes Are and Aren't

A black hole is an object that has so much mass its internal gravity is too great for even a giant nucleus to hold itself together. All of the particles within it continue to collapse into a point, also known as a singularity. This singularity has mass, but no volume; it is just a spot in space. It is this lack of volume that makes it dangerous.

The force of gravity felt between two objects depends on three variables: the mass of one object, the mass of the other object, and the distance between them. The closer the two objects get, the stronger the force of gravity between them. This is why gravity is always strongest when the objects are in contact, and the distance between the centres of the two objects is as small as it can get. In the special case in which at least one object is a planet, the force of gravity is strongest when the second object is on the planet's surface. Imagine Earth is the first object and you are the second object. If you were to start digging into the Earth, traveling down into and through the molten mantle, you would notice that gravity steadily decreased as you got closer to the core.[1] This is because much of the Earth's mass is "above" you as you tunnel, and the gravitational force from that mass is pulling you upwards. Now imagine the Earth has collapsed into a black hole. It has no surface that you can reach, so getting closer and closer just means the force of gravity gets stronger and stronger.

The stronger the pull of gravity, the harder it is to escape. The faster you are traveling, the better your chance at escape. (Rockets can escape Earth's gravity to visit the moon and other places, but people can't escape Earth's gravity by running up a hill and jumping as high as they can.) The key to leaving a planet is reaching "escape velocity," the speed at which one must travel away from the planet's core in order to escape its gravity. A black hole is an object with no volume, which means there is no limit to how close one can get to it. As a result, one can get close enough to it that the escape velocity at that point is greater than the speed of light. The point at which escape velocity equals the speed of light is known as the "event horizon."[2] As we know, the speed of light is woven into the fabric of our spacetime. The geometry of black holes gives them their name: if the "dent" made by the gravity of an object is so steep that not even light can escape, then for all intents and purposes, it punches a "hole" in the fabric of reality. In actuality, the fabric of spacetime still exists within the black hole, but it might as well vanish as far as those outside the black hole are concerned, as the interior of the black hole can never be observed.

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[1] This is assuming you notice anything other than the intolerable heat.

[2] The etymology of this name goes back to the mind set from earlier lessons: relativity doesn't focus on objects at locations in space, but at events that occur at locations in spacetime. Just as one cannot see objects past the horizon on Earth, one cannot observe events that take place behind the event horizon of a black hole.

It is a common belief that, should our Sun be replaced by a black hole with mass equal to our Sun, our entire solar system would be swiftly sucked into this black hole and crushed beyond recognition. This could not be further from the truth. The key to the unusual effects of a black hole is proximity; if you aren't close to the event horizon, you don't notice a difference. If our Sun were to be replaced with such a black hole in some instantaneous manner, Earth's orbit would be completely undisturbed. The lack of incoming energy from fusion would still doom the human race when photosynthesis stopped and temperatures dropped, but it would be a slow and agonizing planetary death instead of the swift and exciting death predicted by underresearched stories. When you do approach the event horizon closely enough, you witness a number of unusual phenomena.

1.3 Black Holes Have No Hair

There are a number of quantities that can be tracked and conserved when studying most objects and events. For example, if car A contains a certain number of electrons, and car B has a different number of electrons, the total number of electrons remains the same after the two cars collide in the street. One can count electrons after the fact and figure out how many electrons each vehicle started with.[3] With a black hole, that and other information is lost. In fact, when studying a black hole, only four pieces of information about the matter that forms the black hole are preserved: the total mass-energy of the objects and particles involved, the net electrical charge of those objects, the net magnetic charge[4] of the objects, and the net angular momentum[5] of those objects. It is said that "black holes have no hair," using the metaphor that the lost information is equivalent to hair. It's a strained metaphor.[6]

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[3] I don't claim it's easy to count them, only that it is conceptually possible.

[4] Although theoretically possible at quantum mechanical levels, magnetic monopoles (north poles without south poles or vice versa) have never been observed. In theory, if they do exist and fall into black holes, these quantities will survive.

[5] Angular momentum is a combination of an object's geometry and rotation speed. A black hole retains this combination, but isolated information about either the geometry or rotation speeds of the objects the black hole is made of is lost.

[6] If the metaphor is so strained, why is it in use? Purportedly, the scientists who proposed the vocabulary did so because several non-English languages translate the term "black hole" into a term that is also used as a slang term that refers to a part of the female anatomy. They thought it was amusing to force that phrasing upon scientists working in those languages. This kind of attitude may be one of the reasons the field of physics is still so male-dominated today. The gender gap is smaller than it used to be, but it is still far too wide.

1.4 Hawking Radiation

One of the reasons Stephen Hawking is so highly regarded amongst the physics community is that he was one of the first to examine the combined implications of quantum mechanics and general relativity, and in doing so he made an astonishing discovery: black holes evaporate.

It is a well known fact of quantum mechanics that "virtual particles" appear and disappear frequently.[7] Hawking was the first to examine what happens when these particles appear and disappear near a black hole. It is possible that one particle falls into the black hole, while the other escapes. These "virtual particles" typically annihilate with each other and disappear to conserve the total energy available. When one is captured by a black hole, it gives the black hole a *negative* amount of energy. Thus, the other particle can continue to exist as a real particle, and the black hole *loses* energy. The smaller the black hole, the faster this takes place. This is what has some people in a panic about particle accelerators: they fear that the creation of tiny black holes will somehow threaten the planet, either because of the energy released by evaporating black holes, or because the tiny black holes themselves are going to somehow consume the planet.

We have already discussed the latter fear; black holes don't suddenly suck everything up. This is a very good thing, as all elementary particles (electrons, the quarks protons and neutrons are made of, etc.) have zero volume, meaning every subatomic particle in existence *is* a black hole. Not big ones, but black holes nonetheless. As for the former fear, energy is still conserved, so a black hole that is made out of six high energy particles can only release the energy contained in those six particles while evaporating. Thus, it can only release energy in quantities less than or equal to the energy that was already contained by the particle accelerator and detector before the black hole was created.

1.5 Frame Dragging, Event Horizons and Ergoregions

Mass and energy have been tied tightly to the very fabric of spacetime. We know that a moving object creates ripples of gravity in this spacetime. What happens if it rotates rapidly, creating a lot of motion-ripples (gravitational radiation) in a single location?

The answer is a phenomenon known as "frame dragging," and it is not limited to black holes. When a massive object rotates rapidly, it can pull spacetime along with it. If you are a stationary observer positioned above this source of gravity (likely a star) you would not fall straight toward the object. Instead,

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[7] See http://fiziko.bureau42.com/b42_qm_summer_school_2010.pdf, chapter four.

you would be “dragged” around the object in the same direction it is rotating as you fall, which is why the phenomenon is called frame dragging. This is the phenomenon that explains the unusual nature of Mercury’s orbit, first mentioned way back on the first page of our first lesson. Mercury’s orbit rotates around the Sun because the spacetime around the Sun rotates *with* the Sun, dragging our solar system’s closest planet right along with it. This effect is most pronounced near the object that is rotating.

Perhaps the most bizarre phenomenon surrounding black holes takes place inside the event horizon. When an observer falls inside the black hole, spacetime gets so twisted that the four directions of space and time get twisted around. Once past the event horizon, an observer is just as free to move through the time dimension as we typically can move through space. The tradeoff is that the “down” direction takes on the position of the imaginary axis, meaning the observer is completely unable to control his or her inevitable descent towards the singularity at the centre of the black hole.

Because the effect of frame dragging is more pronounced as you get closer to the object, it becomes a significant effect when a black hole rotates. In fact, many black holes can rotate so quickly that they form “ergoregions” around them. In these regions, spacetime is being pulled so strongly that it becomes impossible to rotate around the black hole in the direction opposite to its rotation. If the black hole rotates clockwise (from your perspective) then it is impossible to form a close counterclockwise orbit around that black hole. Furthermore, the transformation and replacement of the “down” and “time” directions takes place within the ergoregion instead of behind the event horizon. When one falls through the ergoregion of one of these rapidly rotating black holes, and crosses the event horizon within, the transformation is reversed: within this black hole, time and space resume their normal roles! In fact, it is possible that the entire observable universe is contained within a supermassive, rapidly rotating black hole.

2 Wormholes

With the discovery that mass can warp spacetime, punching holes in it, scientists and science fiction writers got excited again. Perhaps it was possible to circumvent the light speed limit in some way: instead of propelling an object faster than light, one could bend and twist space until the destination is much closer, and then just move there at speeds less than light.

The first scientific suggestion related to this idea is commonly referred to as the “wormhole.” Formally named the Einstein-Rosen bridge (as Albert Einstein and Nathan Rosen first proposed it in 1935 as a way to bridge two points in spacetime) the idea is that a tunnel, or hole, is created in spacetime by two black holes. Each black hole pulls the fabric of spacetime out of shape into an escapable region. It was proposed, however, that two such black holes, properly aligned, could create a tunnel that connected two far distant points in spacetime. Unfortunately, 27 years later, John Wheeler and Robert Fuller showed that such a body is unstable, and would collapse as soon as any mass or energy entered, turning the wormhole back into a pair of black holes before it could come to any practical use.

In 1988, Kip Thorne and Mike Morris showed that stable wormholes are only possible if some sort of exotic matter is held in place at the opening. While this does give some hope for faster than light travel, it is not much hope. The type of matter proposed has properties that have never been observed in our universe, and there have been no effective methods proposed for keeping that matter in place. Try as we might, it appears that the universe simply does not allow time travel to happen in the science fiction sense.

3 Schwarzschild Geometry

At the end of the last lesson, we assembled (but did not derive) a metric that approximated Newtonian behaviors and demonstrated that such a metric necessitated that objects fall under gravity. It was Karl Schwarzschild who, in 1915, found solutions to Einstein's equations that could be formally derived and possessed these properties. The metric he discovered is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{c^2 r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Examination of this metric led to the discovery of black hole theory. In the case in which

$$r = \frac{2GM}{c^2}$$

the metric reduces to

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \infty & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

which is something of an issue for the g_{00} and g_{11} entries. Is this a singularity of the geometry, or simply of the coordinate system? After all, the traditional spherical coordinate system has singularities at $\theta = 0°$ and $\theta = 180°$. Going back to our model of the Earth, if one is at either the north or south poles, you have a singularity of the coordinate system as there is no unique line of longitude (ϕ) to describe that location. Nonetheless, the location is perfectly valid, and the problem is with the coordinates, and not the space itself. An appropriate set of coordinates was discovered in 1960, known as Kruskal-Szekeres coordinates, and it eliminates the apparent issue here with the definitions

$$\begin{aligned} u &= e^{\frac{c^2 r}{4GM}} \cosh\left(\frac{c^3 t}{4GM}\right)\sqrt{\frac{c^2 r}{2GM} - 1} \\ v &= e^{\frac{c^2 r}{4GM}} \sinh\left(\frac{c^3 t}{4GM}\right)\sqrt{\frac{c^2 r}{2GM} - 1} \end{aligned}$$

for $r > \frac{2GM}{c^2}$ and

$$\begin{aligned} u &= e^{\frac{c^2 r}{4GM}} \sinh\left(\frac{c^3 t}{4GM}\right)\sqrt{1 - \frac{c^2 r}{2GM}} \\ v &= e^{\frac{c^2 r}{4GM}} \cosh\left(\frac{c^3 t}{4GM}\right)\sqrt{1 - \frac{c^2 r}{2GM}} \end{aligned}$$

for $r < \frac{2GM}{c^2}$. These definitions share the metric

$$g_{\mu\nu} = \begin{pmatrix} -\frac{32G^3M^3}{c^6 r} e^{\frac{-c^2 r}{2GM}} & 0 & 0 & 0 \\ 0 & \frac{32G^3M^3}{c^6 r} e^{\frac{-c^2 r}{2GM}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

where v is the 0th coordinate. Note that u and v are both unitless in this definition: the required units are “carried” by the metric components, which now all have units of length squared. Many find this aesthetically pleasing, as all components of the metric now have consistent units, and all components of all four vectors would also now have consistent units.

These coordinates handle the coordinate singularity, but they have some behavior consistent with the Schwarzschild metric which is surprising. What happens when $r < \frac{2GM}{c^2}$? In both the Schwarzschild and Kruskal-Szekeres metrics, something strange occurs: $g_{00} > 0$ and $g_{11} < 0$ under these conditions. In other words, the time direction we are used to becomes as easy to navigate as space, but the *radial* direction that carries an object towards $r = 0$ marches inevitably forward! Once an observer gets too close to the point $r = 0$, it becomes an inescapable trap. This is a black hole, and the surface described by

$$R_S = \frac{2GM}{c^2}$$

is known as the event horizon. The distance R_S is known as the Schwarzschild radius. It should be noted that the event horizon is only spherical in the case where the mass M is not rotating.

How dangerous is this? Well, for a body like our Sun, with $M = 1.98892 \times 10^{30}$kg, we have $R_S = 2953.25$m. This is over 235000 times smaller than the Sun's current radius, so we need not worry about our Sun becoming a black hole unless it is somehow compressed to under $\frac{1}{1.306\times10^{16}}$ times its current volume. Notice also that the gravity of the black hole behaves just like the gravity of a normal object for $r \gg R_S$, which is the behavior we would see in Earth's orbit if our Sun were replaced by an equally massive black hole.

3.1 What is at $r = 0$?

We have seen that the radial dimension takes on a timelike nature within the Schwarzschild radius. Thus, one cannot resist an immediate forward crush through this dimension. As a result, any mass within the Schwarzschild radius travels directly to that point in the spacetime. Furthermore, attempting to describe this point in terms of either metric results in division by zero. This is the *singularity* at the centre of a black hole. All mass within the black hole finds its way here. There is no way to distinguish between the different contributions to this singularity: hence the thought that "black holes have no hair." The only quantities which survive the crush are the total mass-energy, net electrical charge, net magnetic charge and total angular momentum. Thus far, we have ignored angular momentum, as it is not a part of the Schwarzschild geometry.

4 Kerr Geometry

The Kerr metric is one in which angular momentum is not necessarily zero, and the black hole is permitted to rotate. If the black hole carries angular momentum J, then the Kerr metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -\frac{\Delta - a^2\sin^2\theta}{\rho^2} & 0 & 0 & -\frac{2Mra\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2Mra\sin^2\theta}{\rho^2} & 0 & 0 & \frac{\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta \end{pmatrix}$$

where

$$\begin{aligned} a &= \frac{J}{cM} \\ \Delta &= r^2 - \frac{2GMr}{c^2} + a^2 \\ \rho^2 &= r^2 + a^2\cos^2\theta \end{aligned}$$

This contains a rather surprising feature: there are non-zero components outside the diagonal of the metric! These terms are directly proportional to a, which is directly proportional to the angular momentum J of the black hole. (Notice also that, when $a = 0$, this metric reduces to the Schwarzschild metric.) These non-diagonal terms represent frame dragging: if an object follows a geodesic path within a particular reference frame, then that reference frame is forced (through the $g_{t\phi}$ terms) to rotate around the source of mass in the same direction that the mass is rotating.

Imagine a particle which is initially at rest, such that $p_\phi = 0$. This has no angular momentum in its one-form. If we look at the four-vector describing its motion, we find that

$$p^\phi = g^{\phi\mu}p_\mu = g^{\phi\phi}p_\phi + g^{\phi t}p_t = -\frac{2Mra\sin^2\theta}{\rho^2}p_t$$

As $p_t = -mc^2$ in the rest frame of the particle, we have

$$p^\phi = \frac{2Mmra\sin^2\theta}{\rho^2} \neq 0$$

indicating that a particle must move in this spacetime. The angular velocity ω is given by

$$\omega = \frac{d\phi}{dt} = \frac{g^{\phi t}}{g^{tt}}$$

Now, in the matrix forms, $g^{\mu\nu} = (g_{\mu\nu})^{-1}$, and it is the version with the lowered indices which we have defined. Thus,

$$g^{\phi t} = -\frac{g_{\phi t}}{g_{\phi\phi}g_{tt} - {g_{\phi t}}^2}$$

and

$$g^{tt} = \frac{g_{\phi\phi}}{g_{\phi\phi}g_{tt} - {g_{\phi t}}^2}$$

such that

$$\omega = -\frac{g_{\phi t}}{g_{\phi\phi}} = \frac{2Mra\sin^2\theta}{\Big(\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\Big)\sin^2\theta}$$

Notice that $\omega = 0$ when $a = 0$, as with the Schwarzschild radius. For large enough J, a begins to dominate. As one approaches the event horizon of a rapidly rotating black hole, there becomes a region in which p^ϕ can never take on a negative value: even a photon which is incident in the opposite direction to the rotation of the black hole is forced to orbit around the black hole in the direction of the black hole's motion.

This phenomenon occurs around anything which has mass and angular momentum, including our Sun. As the Sun is a fluid, there is no simple expression for its angular momentum. It has been shown, however, that this frame dragging is exactly what is needed to account for the seeming incongruities in Mercury's orbit which were known before Einstein developed the theory of relativity. This proved to be one of the greatest successes of the general theory of relativity.

5 Hawking Radiation: The Process

It is known that conservation of energy can be violated for very short periods of time. In other words, the Universe can cheat if it doesn't get caught. Stephen Hawking was the first to publish an analysis of what happens when this occurs near an event horizon.

The specific constraint on energy and time fluctuations comes from the Heisenberg uncertainty principle:

$$\Delta E \Delta t < \hbar$$

where $\hbar$ is a (small) known constant.

We now know that energy and mass are equivalent. Thus, two particles with mass m which appear in a quantum fluctuation represent a certain minimum amount of energy. Normally, these particles collide with each other and annihilate before the amount of time

$$t = \frac{\hbar}{2mc^2}$$

has elapsed. What happens if one falls through the event horizon? The other cannot annihilate with it any more; even if it did fall through the event horizon, the r coordinate is now timelike, so they could never intersect in space until they reach the singularity. This may take more time than is indicated. Hawking showed (through the mathematical mechanisms of quantum field theory, which fall far, far beyond the scope of this text) that the particle which falls into the black hole represents a negative amount of energy. The other particle drains enough energy from the black hole to become a real particle, possibly escaping the gravity well completely. The net result for the black hole is a loss of energy. The smaller the black hole, the faster this process will take place. Black holes with mass around that of stars will survive longer than the Universe, but black holes with small masses evaporate very rapidly. This is the process that terrifies the partially informed about particle accelerators. Yes, if a black hole on those scales is created, it will evaporate quickly with a high energy intensity. The total energy, however, is still tiny; it only has the energy it was created with. As particle accelerators ensure such collisions occur within particle detectors, there is nothing to worry about; this burst of energy will take place at a location in space and time that is within a device designed specifically to safely contain energy on those scales or greater. This doesn't even touch on the fact that the conditions to create those black holes are so unlikely that they probably will not happen within the lifetime of the hardware anyway.

6 Einstein-Rosen Bridges

In 1935, Albert Einstein and Nathan Rosen studied the possibility of Schwarzschild geometry existing without the crush of a black hole. They developed the concept of an Einstein-Rosen bridge, commonly known as a wormhole. The concept is simple: if a black hole punches a hole in space time, and spacetime can curve around on itself, then what happens if two black holes line up on two sides of space time? One could facilitate warp drive by going a shorter route through two black holes, right?

Wrong.

The two black holes in question can only connect in this manner if certain conditions are met:

- They must be connected at the moment of creation.
- Their geometry must be identical.
- They must be perfectly aligned right from the start. If they rotate, the angular momenta must have identical magnitudes but opposite directions to maintain this alignment.

These conditions are so exacting that we cannot expect black holes to be a naturally occurring phenomenon. Identical black holes with identical mass are hard enough to construct, but when you also factor in the simultaneous creation and perfectly planned positioning, it is clear that this will not happen at random. This, in itself, does not eliminate the possibility of warp drive; none of these conditions prohibit perfectly planned artificial construction. That comes into play from the final condition: the geometry must remain identical. If you drop in a mass m in one side of the wormhole, you must drop an identical mass m in the other side at a precisely identical moment. These must be identical in all respects, so if you are dropping in a space vehicle with mass m (including crew) you must have a mass with identical geometry on the other side, which effectively means a ship made out of clones of members of the original ship who are so identical that they would have the same brain patterns and would make all the same decisions with all the same resources.

To this point, it is all possible, though extremely unlikely. The difficulty come in examining the trip: despite some popular science fiction, there is no such thing as a “white hole” on the other side spewing things out. What falls into the black hole stays in the black hole. If you align everything perfectly so that you enter the wormhole without breaking it, you still end up getting dragged inexorably towards the singularity, completely unable to escape. In other words, your experience falling into a wormhole will be *identical* in every respect to the experience you would have falling into a black hole. This is not warp drive: this is astonishingly complicated and expensive suicide.