Paradoxes of Relativity
Version A: Full Math

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Contents

1 Introducing the Paradoxes

When Einstein's theory of relativity was first published, its counter-intuitive implications were met with extreme levels of doubt and skepticism. Early researchers actively sought out logical contradictions in the framework that could be used to justify a wholesale rejection of the theory so that it might be replaced with something more aesthetically pleasing. Some of these "paradoxes" are described here. Study of the paradoxes is remarkably illuminating, as it forces the student to directly confront intuitions and prejudices developed from years of working with Newtonian mechanics.

2 Length Contraction

The length contraction paradox is one of the earliest paradoxes. Imagine you have a sports car that is 5 meters long, and your garage is only 4 meters deep. Well, if "moving objects contract," as Einstein's theory predicts, then you should be able to fit your sports car in your garage, provided you are moving quickly enough that the moving vehicle contracts to a length that is less than 4 meters. Everything is fine until you stop the vehicle.

The paradox comes from the equivalence of all reference frames. From the perspective of the driver, the car is stationary, and it is the garage that is in motion. In that case, the 5 meter car is being approached headlong by a garage that is less than 4 meters long: the garage is still smaller than the car. An observer standing near (or in) the garage, at rest with respect to the garage, concludes that the car fits in the garage; an observer in the car concludes that the car does not fit inside the garage. Now, the car is either in the garage or it is not, in all reference frames. So, which is it? This was one of the "thought experiments"[1] that seemed to cause problems for Einstein's theory.

2.1 Relativity of Simultaneity

The key to resolving this paradox lies in something known as the *relativity of simultaneity*. Two events are considered simultaneous if they happen "at the same time." It is not immediately obvious to most people that this concept depends greatly upon how we measure time. If time is measured differently by two observers, will they agree on whether or not events are simultaneous?

[1] A "thought experiment" is an experiment one thinks about but doesn't actually do. This causes a dramatic improvement in lab budgets, allows a tremendous opportunity to explore implications of a theory, and severely limits the ability to make quantified observations to verify a theory. Thought experiments are best used to design experiments to be done physically later.

Let us examine what we mean when we say two events happen "at the same time." This means that information about these two events *reach the observer* at the same time. To pick a more mundane example, picture two different patrons at an outdoor concert in a very large venue. The stage has a large screen at the rear with live video of the vocalist, and most of the speakers projecting sound to the audience are at the front. The patron near the stage believes that the sound and image are simultaneous, as the lips and lyrics are synchronized on screen. The patron at the rear of the venue believes that the sound is out of synchronization with the video because the light from the screen brings that patron information about the video a short time before the sound from the speakers reaches the patron. This is true in Newtonian mechanics as well, simply because light is faster than sound, so that information arrives faster. They do not appear to be simultaneous.[2]

In relativity, we can pretend all information travels at the speed of light. A somewhat analogous situation arises in this case. Events are simultaneous for an observer if the information about those two events arrives at the same time. Since all information travels at the speed of light, this amounts to saying they traveled the same distance to the observer. This is easy enough to imagine when nothing is in motion: the observer is exactly centered between them.[3] What if the observer is in motion? What if one or more events are in motion?

Imagine that the events take place at locations A and B. To an observer who is at rest relative to those locations and is exactly half way between them, the events are simultaneous. Now imagine an observer in motion from location A to location B. This moving observer will only consider the events to be simultaneous if she or he is the same distance (equidistant) from both locations when information about the two events arrives. So, if the moving observer is exactly on top of our stationary observer when the information arrives, both observers will agree that the events are simultaneous. However, if the moving observer is not in this symmetric position, then information about the two events will arrive at two different times, and the moving observer will not consider the events to be simultaneous.

[2]This analogy, of course, derives from limitations of human perception. If we could accurately perceive time in smaller intervals, the patron closer to the stage would also recognize that the events are not simultaneous. The only observer who could say the two events are simultaneous would be the observer who eyes and ears are pressed up against the face of the vocalist. This is not a viewing location recommended to those who want to watch the entire concert without being dragged off by employees of the venue's security force.

[3]Technically, the observer could be anywhere on the infinitely large plane between the two events. Imagine that the two events that occur are centered along the outside edges of the pages of an open book. The observer could be anywhere alone the spine where the pages meet. This book can be as tall as you'd like, and can be spun around. The observer can be any place the spine can be. This forms the infinite plane.

2.2 Resolving the Length Contraction Paradox

Somewhat contrary to intuition, the definition of simultaneity plays a large role in the length contraction paradox. Upon closer inspection, the paradox arises because the mental images it conjures for the driver involve knowledge about both ends of the car simultaneously. In other words, the transmission of information is considered *instantaneous* instead of delayed, violating the postulates of relativity. The solution to the paradox comes through in a manner that surprised the opponents of relativity: when the car enters the garage, the front of the car strikes the rear garage wall. Information about this collision is then transmitted[4] to the rear of the car. In the time it takes this information to reach the rear of the car, the rear of the car has *already* entered the garage;[5] both observers agree that the car fits in the garage! Both observers also agree that the driver is insane and the car is a write off.

3 Twin Paradox

The equivalence of reference frames also leads to problems with time dilation.

Imagine you are a twin.[6] At present, you are both the same age.

The paradox begins when you load your twin into a rocket ship and launch him or her on a round trip to another solar system a few light years away. Your twin is in motion the entire trip, so as far as you are concerned, time moves slower in her or his frame of reference. When the craft returns, you will be the older twin. Now imagine your twin's perspective. Your twin can consider himself or herself to be at rest the entire trip, so you should be aging more slowly, and your twin will be the older twin upon her or his return. Which of you is older?

[4]The mechanics of transmission are irrelevant to the paradox. In the physical world, the information is transmitted via vibrations in the bonds between atoms and molecules in the car, and is transmitted at less than the speed of light. A lower transmission speed assists in the resolution of the paradox, and will be ignored in the main argument.

[5]This is guaranteed. If you look carefully at the math, the condition that says the car is fast enough for length contraction to make the car fit in the garage from the perspective of the observer in the garage is identical to the mathematical condition that the rear of the car enter the garage before information about the collision arrives. The car fits or doesn't fit when measured by all observers.

[6]If you really are a twin, imagine this is one of the days that you and your twin are getting on each other's nerves.

3.1 Resolving the Twin Paradox

The key to this paradox is the recognition that we are not, in fact, dealing with inertial reference frames. Recall the definition from our previous lesson: an "inertial" reference frame is one in which the origin of the coordinate system is not being accelerated. Acceleration includes direction as well as speed. When the twin in the rocket turns around for the return trip, she or he changes from one inertial reference frame to another. In reversing direction, the pretense of simultaneity is broken, and the world proceeds as usual.

Complete details are available in the version of this lesson that includes the math. The crux of the solution is this: when the twin is rocketing away from Earth[7] both are correct in treating themselves as the "older" twin. If each twin sends signals to the other at regular intervals (such as birthdays) then both twins send more signals than they receive while they are moving apart. When the twin in the rocket reverses course, however, the timing is disrupted. This twin begins to see signals more quickly as he or she approaches their point of origin. By the time both twins are back on their home world, the twin in the rocket has received more signals than she or he has sent, and they agree that the Earthbound twin is older.[8] This does leave one more question, though: what is the age of the twin who made the round trip? For that, we measure the *proper time*. Proper time is not "more correct" than any other time, despite the instinctive reaction to the name. It is the time as measure in the rest frame of the moving twin. This is the amount of time that lines up properly with that twin's natural aging process.

These are not the only "paradoxes" of relativity that one can find. Thankfully, all are resolvable: every seeming paradox proposed to date has been found to be reconcilable, and most often "arise" by applying our normal, day-to-day intuitions which are actually in error, rather than the theory of relativity itself.

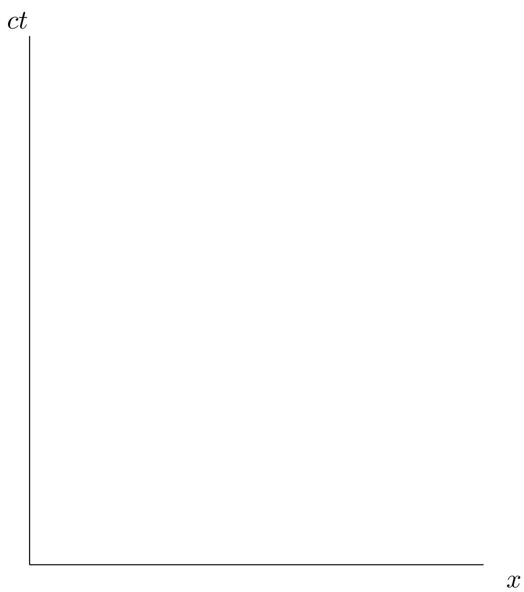
4 Length Contraction Paradox: Mathematical Solution

There are two possible approaches to solving this problem. One is to do three pages of algebra, calculating the observed lengths of both objects in both reference frames and then comparing with the time lag. The other way is to use a single Minkowski diagram. We shall use the latter approach.

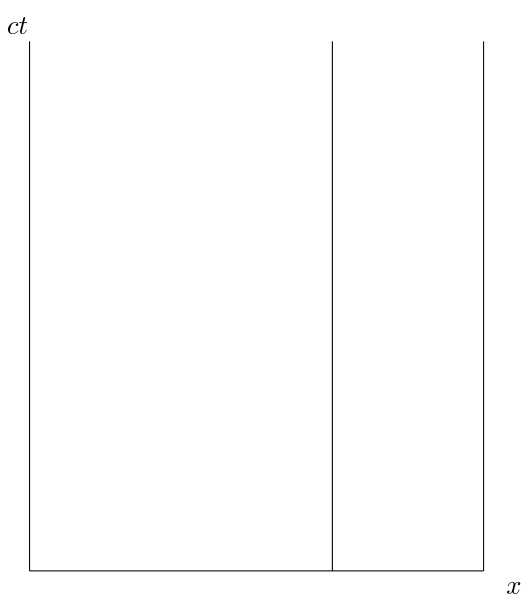
[7] Assuming our readers are on Earth. If you are reading this on another planet, please e-mail me at fiziko@gmail.com and let me know!

[8] This is punishment for sending your twin on a long term interstellar trip, I suppose.

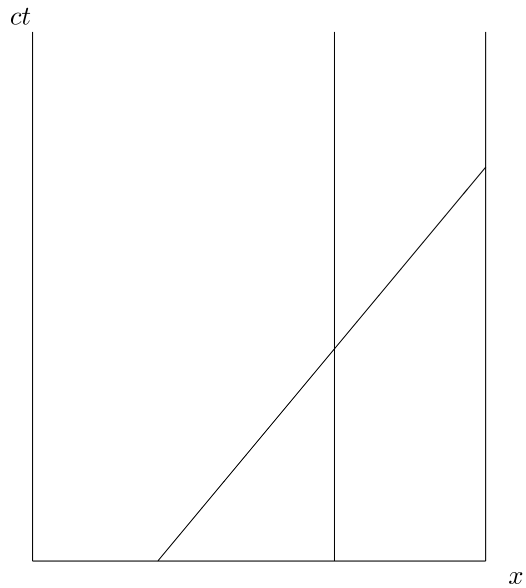
We begin by drawing our axes, taking the garage as frame S and the car as being at rest in frame S':



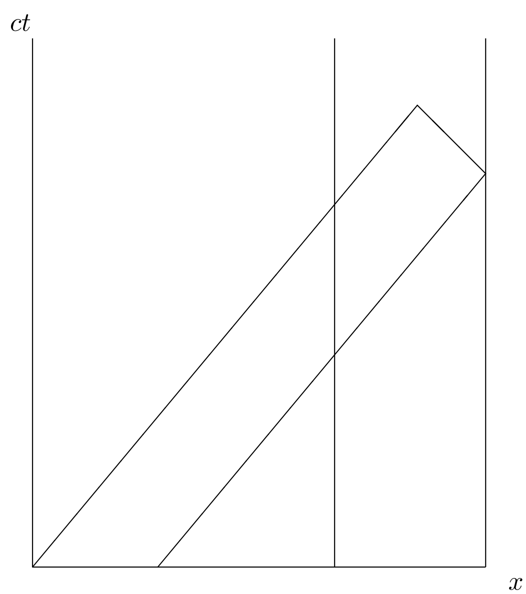
Let the garage be two units wide:



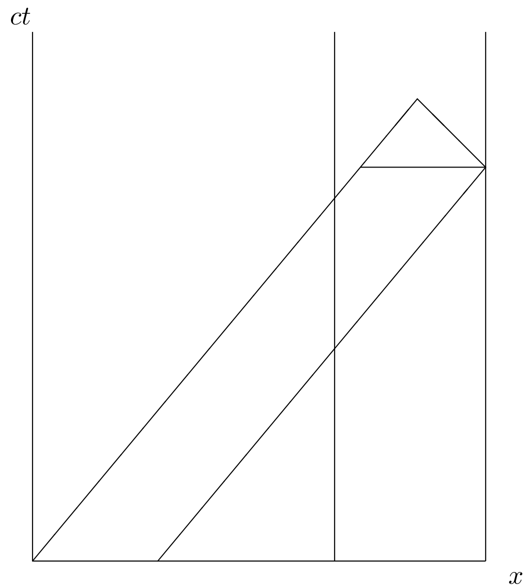
Now we let the car be three units wide in frame S', and traveling at $v = \frac{5}{6}c$. This contracts it to $\frac{\sqrt{11}}{2}$ units long in frame S'. We plot the front of the car up to and including the moment it hits the back wall of the garage:



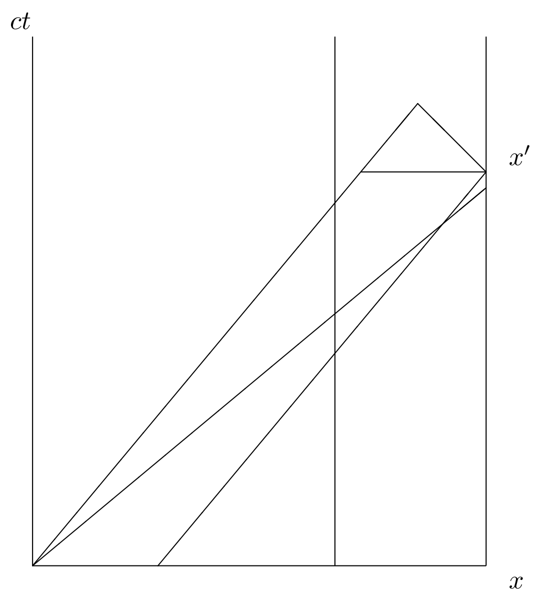
We can plot the rear of the car the same way. Simultaneously, we plot the "information" about the collision of the front of the car with the back of the garage, and run both lines until they intersect.



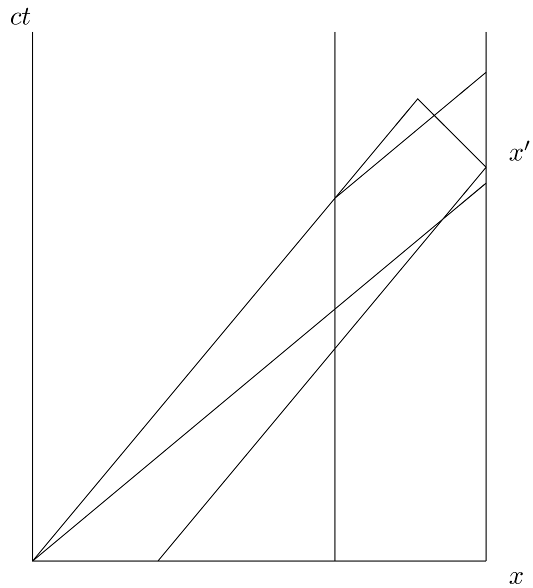
From the perspective of reference frame S, any points that occur along lines parallel to the x axis are simultaneous (same ct value), so a short horizontal line can be used to show that the entire car fits in the garage from that reference frame:



At the point of intersection, the back of the car is inside the garage from the perspective of reference frame S. From the perspective of reference frame S', the back of the car is always sitting on the ct' axis, so we simply add the x' axis:



Now, to check frame S', we look at where the rear of the car "perceives" the front of the car to be at the point of collision. We do this by creating a line parallel to x' coming from the point at which the rear of the car and the front of the garage are in identical locations in space.



Note that this line is "below" the point of intersection between the rear of the car and the information about the collision at the front of the garage. Thus, due to the relativistic version of simultaneity, the rear of the car enters the garage *before* the collision in both reference frames.

5 Twin Paradox: Mathematical Solution

The key to resolving the twin paradox is to calculate the time elapsed in each half of the trip separately. This will automatically cover the change in reference frames.

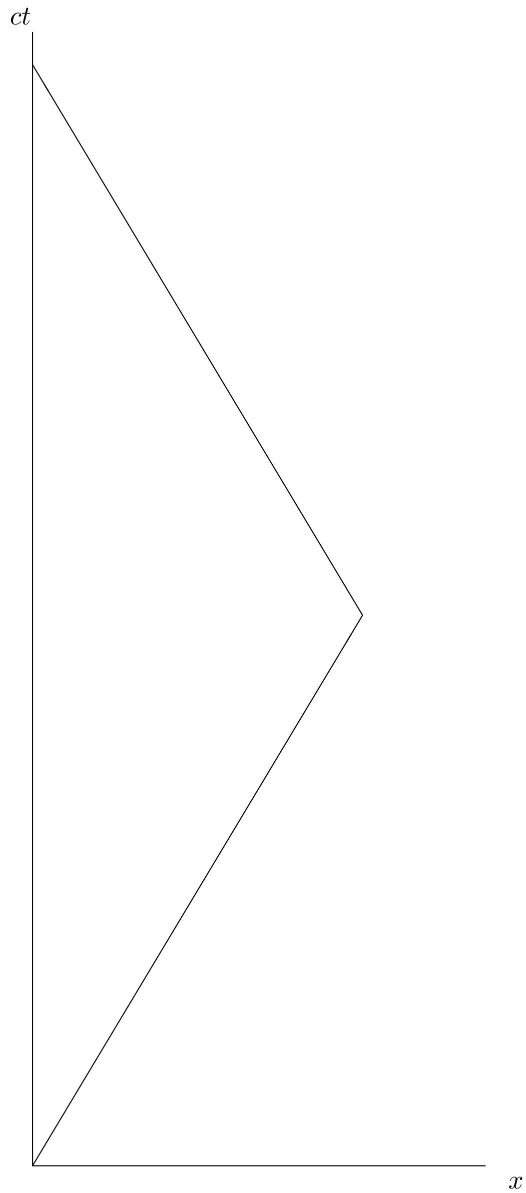
There are three events in time of relevance. The first is the event of launch, when one twin remains on Earth and the other does not. At this point, we can set $ct = ct' = x = x' = 0$lyr by choice. (We will be measuring distances in lightyears and time in years for this example.) The second is when the twin in space reaches the destination, and the third is when the twins are together on Earth once more.

Imagine Tweedledee is the twin who stays on Earth, while Tweedledum is the twin launched into space.[9] Tweedledum is being launched on a round trip to Alpha Centauri, 4.37lyr away, at the speed of $v = \frac{3}{5}c$.

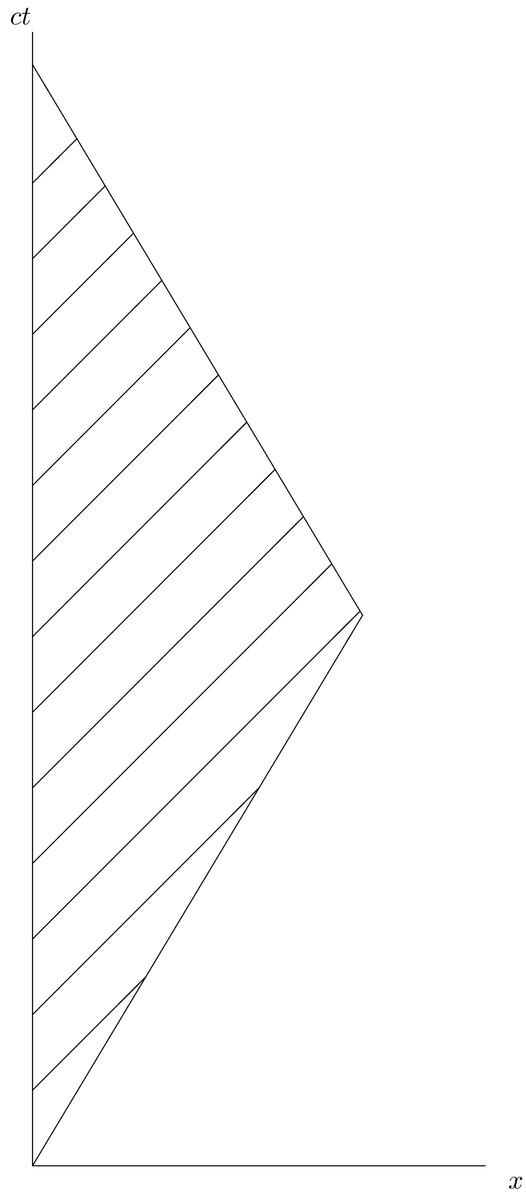
[9]The author hopes Mr. Carroll would not object to the use of his characters for educational purposes. Given the number of mathematical parables in *Alice in Wonderland*, this seems somewhat likely.

Once again, this situation is most effectively examined using a Minkowski diagram. We begin with the axes representing Tweedledee's reference frame.

Tweedledum travels the 4.37lyr to Alpha Centauri and back at speed $v = \frac{3}{5}c$:

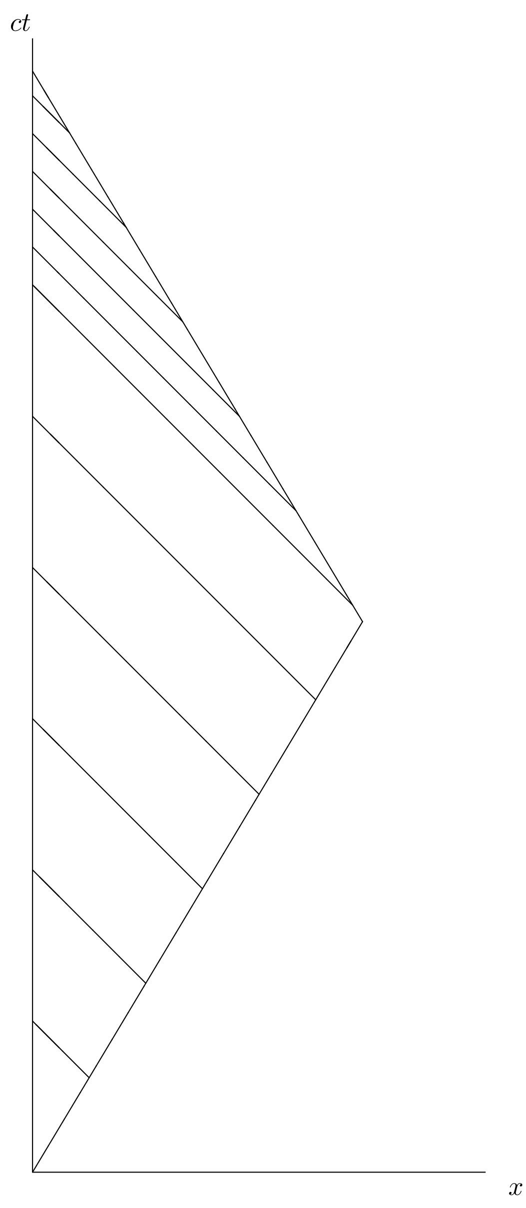


Recall that time dilation is an effect produced by the light speed delay in transmitting information. If Tweedledum watches Tweedledee, he can mark the years passed for Tweedledee as the light from the end of each year catches up to him. Marking those years, we see the following:



Note that time accelerates from Tweedledum's perspective on the return trip. In this trip, $v = -\frac{3}{5}c$, where the sign change represents the change in direction. If you look only at the section in which the twins move apart, each says the other is the younger twin, and each is justified in saying so within his own reference frame. Tweedledum sees Tweedledee age almost three years on the trip to Alpha Centauri, but he sees Tweedledee age nearly 12 years on the trip back. We now need to observe how Tweedledee ages.

If Tweedledee monitors Tweedledum's years in a similar fashion, we see a similar result. From Tweedledum's perspective, Alpha Centauri approaches him at $v = \frac{3}{5}c$, such that length contraction reduces the trip to 3.50lyr, for a trip time of 5.83yr each way. Of course, we need to be sure we know where to mark the years on Tweedledum's path. The axes for the S' frame use different scales than those of the S frames. In this case, knowing that Tweedledum perceives the voyage as a 5.83yr trip each way, we divide Tweedledum's line into segments in proportion to the 5.83yr total, and mark the years at those points.



Thus, once reunited at the same point in spacetime after the trip, both twins agree that Tweedledee has aged 14.6yr and Tweedledum has aged 11.7yr. Thus, Tweedledum's proper time is $\tau = 11.7$yr The change in reference frames at Alpha Centauri for the return trip is enough to reconcile the paradox.