Space and Time
Version B: Mathless

W. Blaine Dowler

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1 Space and Time Before Relativity

Prior to Einstein's work, the importance of both space and time to physics was clearly recognized. They were, however, treated as completely distinct quantities.

On one hand, they had the spatial dimensions. These came in three fundamental directions. The scientist had the ability to choose these directions to some degree. On the surface of the Earth, one could choose North as one direction, East as another, and Up as the third. If one chooses North then one does *not* choose South; mathematically, they describe the same line, since you can "undo" all travel in the direction North by traveling in the direction South. Directions are only "different" if they cannot be completely canceled by motion in the previously defined directions. One could choose Northeast as a direction, since the "East" portion can't be canceled by North, but choosing directions that aren't at right angles to each other usually makes things more complicated than they need to be when working on the math. The directions are connected through rotations: if you rotate through a right angle, the person who was facing North can now be facing East or Up. It is easy to see how they relate to one another, and that close relationship is one of the reasons we have the ability and inclination to define whatever directions are convenient for us. (For example, when working on a ramp, it is more convenient to pick directions parallel and perpendicular to the surface of the ramp than simply North, East and Up.)

On the other hand, there was (and is) time. No clear connection between space and time was apparent. Sure, they could plot graphs using time as one axis and space on another, but that was a technique used to see how systems evolve, and was not considered an indication of any deeper, fundamental connection between the quantities. Time was absolute and the same (invariant) for all observers, while distances were different for observers in motion (though not as different as they are under Einstein; the distance a car traveled was variable under Newtonian mechanics, but the length of the car was not, since the front and back could be measured "simultaneously.")

2 Space and Time After Relativity

With Einstein's relativity, we learn that time is not invariant, and that perceived time intervals depend greatly upon the motion of the observer. Moreover, the manner in which one measures distances depends on time measurements and vice versa. This implies a much stronger connection between the two quantities than was previously believed. This begs a number of questions: if time is that closely linked to space, can it be treated as another direction? If so, why does that direction behave so much differently than space? Why can we not navigate back and forth through time as easily as we do in space? Why can't we simply rotate from a spatial direction to a time direction as easily as we rotate from North to East? These questions opened some very productive floodgates. Physicists were forced to examine their beliefs about geometry, which led to some startling revelations when they explored the abstract discoveries made by mathematicians in the recent centuries.

Mathematicians can always be depended upon to indulge their curiosity, regardless of whether or not their discoveries apply to the "real world." One such avenue was the exploration of the square roots of negative numbers. For centuries, it was believed that negative numbers could have no square roots. In fact, they can, although it was difficult for even mathematicians to accept that fact when they were first proposed. This resistance is what led to the square roots of negative numbers being named "imaginary numbers:" even the mathematicians at the time had a hard time accepting them.[1] Despite their name, imaginary numbers are just as valid as any other numbers. Moreover, mathematicians had found that one can include directions describing imaginary numbers in geometry and arrive at precisely the type of system physicists needed. If *time* were measured along an imaginary direction, then it would be a direction tightly linked to the spatial directions, and yet would still be distinct and unique in precisely the manner required to satisfy intuition. No simple[2] rotation can point someone in an imaginary direction, yet the direction is an integral part of the geometry of reality. This distinction is enough to explain why we cannot control our motion along the time direction: it is a fundamentally different type of direction. Still, this would have been incredibly difficult to accept if not for the discovery of invariants.

3 Invariants

The realization that this mathematical framework described the time direction went hand in hand with another accidental discovery that had yet to be explained, related to the lengths of certain mathematical objects.

In Newtonian terms, moving observers disagreed on the positions of objects, but they always agreed on the lengths of those objects. Relativity threw that convenience out the window, but did so in a way that pulled time in as another direction to be considered in the geometry. Physicists began working with var-

[1] Most high school students have heard of the real numbers. They were named to contrast the imaginary numbers in an act of professional mockery. "You can study those imaginary numbers if you'd like, but *I* work with *real* numbers.

[2] This word is surprisingly important, as we'll see in section 4 on the following page.

ious ways to combine the space coordinates with the time coordinate to see if this idea of an "invariant" length could be preserved. They had discovered a means to make this happen that was remarkably counterintuitive.

If an object is turned to lie along one of the directions chosen for our coordinate system, then it is relatively easy to measure its length: find the coordinates of the two ends, and subtract the smaller number from the bigger number. Since the days of Pythagoras, there was a known method for measuring the length of an object that didn't lie along one of these directions: measure the coordinates of the extreme ends of the object along each of the chosen directions, square them, add the squares, and then take the square root of the answer. The calculation may be somewhat cumbersome without a calculator, but it is useful to know how to do this when one cannot simply turn the object or the measuring device to measure something in the simple, intuitive manner.

In trying to do this with relativity, physicists noticed something very strange. Adding in the square of the time coordinate didn't resolve the issue: both time and distance were smaller numbers in a moving reference frame. The strange and startling discovery was that these reductions could cancel each other out if subtracted! In other words, if one added the squares of the lengths of the object along the space dimensions, but then *subtracted* the square of the length along the time direction, the answer one got was the same for *all* observers. Quantities that are the same for all observers are known as *invariant* quantities.

This idea was shocking and counterintuitive, but completely consistent with treating time as an imaginary number. Recall from the previous section that imaginary numbers were discovered as the square roots of negative numbers: if one squares an imaginary number, the answer is negative. Adding a negative number is mathematically identical to subtracting a positive number. The implications of relativity get crazier and crazier, and yet somehow remain entirely consistent with each other and with experimental results. Slowly but surely, Einstein's seemingly insane notions were gaining more and more support.

4 Minkowski

Hermann Minkowski laid much of the foundation for this combination of space and time when he addressed the attendees of the 80th Assembly of German Natural Scientists and Physicians[3] on September 21, 1908.[4] His speech began

[3] No, the term "physicist" was not in use yet. Should you travel back in time by more than a century and develop some sort of medical ailment, be sure you know the true expertise of any physician you encounter.

[4] Minkowski was 44 years old at the time of his address, and would die of appendicitis in January 1909.

as follows (after translation):

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

Minkowski's address then continued to lay the foundations for the work done earlier on in this lesson, though it should be noted that the usefulness of imaginary numbers had yet to be realized; Minkowski subtracted the square of time, not because he treated time as imaginary, but because it was already known that this was an invariant quantity. Still, he laid a geometric foundation that is incredibly useful today: the space-time continuum.

The difficulty in graphing problems in relativity is that time can be treated as an imaginary fourth direction. We can graph two dimensions easily, and can approximate graphing three dimensions by "projecting" them onto a two dimensional page. Graphing four dimensions cannot be done on two dimensional paper with pen or pencil.[5] Minkowski found a means to graph space and time together without losing the unique qualities of time. He realized that we could not, as is traditional, keep all of our axes at right angles. These so-called Minkowski diagrams formed a picture to go along with the concepts, and opened the gates to allow physicists to envision the situations in a useful way. They could graph situations as measured by two different observers on a single graph, making their work far more efficient. This also added another tool to the toolkit.

With space and time this closely connected, and with multiple observers graphed on the same diagram, an even more startling discovery was made: there *is* a rotation that allows one to rotate from a spatial direction into a time direction! This rotation manifests itself physically in an unusual fashion: accelerating to a new speed is mathematically identical to rotating with respect to the time access. The "rotation" exists, and we do have the ability to implement it, but the imaginary nature of the time direction means it "looks" completely different than a spatial rotation to us.[6] We still cannot travel backwards in time, but Minkowski's work laid foundations to explain that as well.

[5] Approximations can be made by creating multiple three dimensional graphs, each at a different time, but this is cumbersome and makes the time evolution of the system more difficult to see from the graph alone.

[6] It has been proposed that there are life forms which perceive time no differently than they perceive space, though no concrete evidence exists. It has also been proposed that said life forms live inside an artificial wormhole and are worshipped by the inhabitants of a nearby world that look identical to humans aside from their noses, so the science involved is highly questionable.

4.1 Causality

It is an axiom[7] of science that cause precedes effect. Relativity seems to support this concept.

Imagine two events in the space-time continuum. One happens in a given place and time, and the other happens in a different place at a different time. One can calculate the invariant distance between these two events, in the manner described above. These calculations involve both positive and negative quantities, and the results can be positive, negative or zero. We say that the events are *time-like separated* in one case, *space-like separated* in another, and *light-like separated* in the third. Time-like separated events are those which can be connected by cause and effect, while space-like separated events cannot. Light-like separated events form the border between these two categories.

Let us examine explicit examples. Imagine one drops a rubber ball on the floor, and you consider the two events in question to be the first time it hits the floor, and the second time it hits the floor (after bouncing.) These events are time-like separated because they happen at the same point in space; it is only time that separates them. The cause-effect relationship is possible and established by observation. Any two time-like separated events find that the square of the distance between them in the time direction is greater than the sum of those in the spatial directions. As a result, information about one event can travel at the speed of light at reach the spatial coordinates of the second event *before* the second event takes place.

Now imagine two people are bouncing balls, one on planet Earth, and the other on the rock formerly considered a planet which is named Pluto. If an observer in a spacecraft half way between them believes they are bouncing their balls simultaneously, then this observer will say that there is a separation in space, but not in time. These events are space-like separated; neither person can have any effect on the other person's ball bouncing activity. There is a difference in space, but not in time. Information traveling at the speed of light can leave one event the instant it happens, but by the time it reaches the spatial coordinates of the second event, the event has already happened. There can be no cause-effect relationship.

In the case of light-like separated events, the cause-effect relationship is possible, but the calculation of the four dimensional distance between the two events works out to exactly zero.

These ideas were not terribly new. What was new is the mathematical implications that seemed obvious in one of Minkowski's graphs: events that are

[7] An axiom is an idea that is accepted as true, but which cannot be formally proven. They typically haven't been formally disproved, either, as that would result in the loss of the axiom.

time-like separated from the perspective of one observer are time-like separated from the perspectives of *all* possible observers. Similarly, space-like separations are always space-like, and light-like separations are always light-like. The implication supports the idea that cause and effect relationships must be preserved in space and time, and that "time travel"[8] is impossible. This idea will be revisited when gravity comes into play in a later lesson.

[8] Here the term "time travel" is used to mean traveling from an arbitrary point in time to another, equally arbitrary point in time, with little or no perceived time lapse between them.