Math From Scratch Lesson 1: Relations

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1 Preface

This is the first part in a rather large project. The goal: to teach all of the math I know in a logical, sequential, and complete order over the coming years (or decades.) I intend to have a maximum of three weeks between lessons. Requests and suggestions are heartily encouraged.

2 Relations - Why Start Here?

In teaching math on a scale of this magnitude, one needs to choose a natural starting point. The most natural starting point is to define what numbers are and what we can do with them. It is more natural to define comparisons between numbers first, and then define numbers themselves later.

3 Relations

A relation is a means of comparing two or more mathematical objects. If the comparison is applicable, the relation is true. If the comparison is inaccurate, the relation is not true. The best known example of a relation is equality, written $=$, which indicates that the mathematical objects to the left of the symbol are somehow identical to the object to the right. This relation will now be formally defined through its four defining characteristics.

3.1 Transitivity

A *transitive* relation R is one which satisfies the following property when used in relation to mathematical objects a, b and c:

Definition Assume that, for a relation R and mathematical objects a, b and c, we know that aRb and bRc are both well defined and true. R is a transitive relation if and only if, when both aRb and bRc are true, then aRc is also true.

In other words, the truth of the relation translates across multiple relations, such that having aRb and bRc together implies aRc. Note that b appears on two different sides of R: this is a *mandatory* piece of the definition. The reader is asked to forgive the author for what is about to come: numbers have not been formally defined yet, but the nature of transitive relations will be made more clear using concrete examples most readers are likely familiar with.

The "less than" relation, denoted $<$, is a transitive relation. If $1 < 2$ is true and $2 < 3$ is true, then $1 < 3$ is true. If the position of b was not relevant in the above definition, then we would find that if $2 < 3$ is true and $1 < 3$ is true, then $2 < 1$ would also be true, but it isn't.

When relations are transitive, we can often take another shorthand. If we know that aRb, bRc, cRd, dRe, $\ldots$ are all true, then we tend to compress them into a long string of statements, such as $aRbRcRdRe\ldots$ to make use of another mathematical shorthand.

The final shorthand in use here is to drop all of the "is true" bits out of these statements. Unless otherwise specified, we always assume the "is true" part of theorem, definitions and statements. It just saves a lot of time, effort, ink and paper to write in the words that most readers would assume are present anyway.

3.2 Reflexivity

Definition A relation R is reflexive if and only if, for every mathematical object a which can be compared using R, the statement aRa is true.

This property is held by equality, but not by the less than relation. For example, $1 = 1$ is true, but $1 < 1$ is false. The "less than or equal to" relation is also reflexive, as $1 \leq 1$.

3.3 Symmetry

Definition A relation R is symmetric if and only if the truth of aRb implies the truth of bRa, for *any and every* pair of mathematical objects a and b.

Notice the inclusion of the phrase "any and every" in that definition. The relation $\leq$ is not considered symmetric because $a \leq b$ and $b \leq a$ are only simultaneously true in the special case of $a = b$. Equality is the relation which is transitive, reflexive and symmetric for every sensible set of mathematical objects which it can be used to compare.

3.4 Antisymmetry

Definition A relation R is antisymmetric if and only if aRb and bRa cannot both be true unless a and b are identical, entirely indistinguishable objects.

If all four of these criteria are met, we have the relation of equality, and write $a = b$. If only the first three are met, we have an *equivalence relation* instead. We will return to equivalence relations in lesson 7.

4 Mathematicians Are A Special Kind of Lazy

Mathematicians are a special kind of lazy: they don't care how much work they need to do today if it will save them work tomorrow, and the day after, and the next day, and the next... Defining shorthand notations for everything serves this purpose well, and also allows for easier communication between mathematicians who speak different languages. The author will define and implement as many of these shorthand notations as possible to keep this project down to a manageable size. Some of the first to be implemented in this lesson and the next are detailed in the following table:

Symbol	**Represents**	**Means**
$\Rightarrow$	"implies"	The truth of the statement on the left implies truth of the statement on the right.
$\Leftrightarrow$	"if and only if"	If any one condition on either side of the symbol are satisfied, then both conditions are satisfied.
$\forall$	"for all" or "for every"	The statement will be true regardless of the value of the variable that follows.
$\wedge$	"and"	And.
$\vee$	"or"	At least one of the conditions around the symbol are true, and possibly both.
$\oplus$	"exclusively or"	Either the condition on the left is true or the condition on the right is true, *but not both.*
$\exists$	"there exists"	There is at least one example, no matter how exceptional or rare, which satisfies the condition on the right.
:	"such that"	Such that

To see how these symbols work, our three above definitions will be rephrased using them:

- **Transitivity**: R is transitive $\Leftrightarrow \forall a \forall b \forall c, aRb \& bRc \Rightarrow aRc$
- **Reflexivity**: R is reflexive $\Leftrightarrow \forall a, aRa$
- **Symmetry**: R is symmetric $\Leftrightarrow \forall a \forall b \; aRb \Rightarrow bRa$

Given sufficient practice, these symbols will appear as clearly as English to the reader, and the mathematical statements get considerably shorter. The author will endeavor to make the transition from the English originally used here to the more compact mathematical notation as smooth as possible over several lessons to help ease the transition.

5 Next Lesson

Now that we have the basic structures needed to compare mathematical objects, we will start to define what the basic mathematical objects are.