Math From Scratch Lesson 2: Elementary Set Theory

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May 28, 2011

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1 Set Theory and Numbers

Now that we can compare objects using relations, we need to start defining what mathematical objects are.

For this, we need set theory.

Intuitively, a set is a collection of things. You may have a complete set of baseball cards, of commemorative Star Trek Royal Dalton chinaware, of Babylon 5 DVDs, of Pez dispensers, and so forth. You may also have sets with no visible connection or relationships to one another, such as a set consisting of ten dollars, two credit cards, a hairbrush, and a lipstick. Collecting objects into sets is a natural process, and one that has a lot of mathematical power.

2 Set Theory - Formalized Notation

In mathematics, sets are enclosed between curly brackets: $\{\}$. Everything between the brackets is said to be a *member* or *element* of the set, and are separated by commas. If a set is defined as $\{a, b, c\}$, then a, b and c are all members of that set.

Constantly writing "a is a member of $\{a, b, c\}$" gets tiring and cumbersome, so mathematicians invented a shorthand notation for this. The symbol $\in$ is both a relation[1] and the shorthand for "is a member of," so "a is a member of $\{a, b, c\}$" is typically written "$a \in \{a, b, c\}$" instead.

One set has no members. This is known as the empty set, and it is written as either $\{\}$ or $\emptyset$. The first notation stresses that this is a set which contains no members. The second looks more like a fancy zero[2] than a set. There is an excellent reason for this, which will be explored in a later lesson.

Sets may also contain other sets. If this were not the case, all of math would collapse for reasons we will see (relatively) shortly.

3 Set Relations

3.1 Subsets

Now that we can define sets, we can start to build the relations needed to compare any two sets. The first relation, $\subseteq$, is a transitive and reflexive relation that is read "is a subset of." This compares sets in a rather natural way: if A and B are sets, then $A \subseteq B$ if and only if every member of set A, without exception, is a member of set B. Formally, using full mathematical shorthand:

Definition Subset: $A \subseteq B \Leftrightarrow (\forall x, x \in A \Rightarrow x \in B)$

The reflexivity of this relation is a side effect of a subtle aspect of the definition: B is in no way required to have a member which is not a member of A. Therefore, $A \subseteq A$ is true for every possible A, as $x \in A$ implies $x \in A$ in the pedantically obvious sense. This is why the symbol appears as it does: it is designed to remind the reader of $\leq$. The straight horizontal line allows the two

[1] The $\in$ relation is always transitive, but neither symmetric or reflexive in all cases.

[2] The author assumes readers do not object to the use of the term "zero" before it is formally defined later.

objects being compared to be equal in both cases, while the shape which opens to the right implies intuitively that the object on the right of the relation may be somehow larger or greater than the object on the right. In the case of sets, B has all the members of A, and may or may not have more.

There is one seemingly ambiguous case: what of the empty set? If a set has no members, is it considered a subset of another set? In mathematics, the answer is "yes." Mathematical formalism of logic defines an untestably nonsensical case as "true," specifically to simplify cases like this. Thus, $\emptyset \subseteq A$ regardless of whatever set A happens to be.

3.2 Proper Subsets

Definition Proper Subset: A is a proper subset of B, denoted $A \subset B$, if and only if the following two conditions are satisfied:

1. A is a subset of B, or $A \subset B$
2. B contains at least one member that is not in A, which can be formally written as
$$\exists y : ((y \in B) \wedge (y \notin A))$$
where we have used the shorthand notations ":" for "such that," $\exists$ for "there exists" and $\notin$ for "is not a member of."

This relation is transitive, but it is not reflexive.

3.3 Equal Sets

We can now use define the equality relation between sets. Two sets A and B are considered equal if they have exactly the same members. In other words,[3]

$$(A = B) \Leftrightarrow ((A \subseteq B) \wedge (B \subseteq A))$$

Compare the definitions of equal sets and of subsets for a moment. Notice that the formalism is completely independent of the order in which members of the sets appear. In other words, $\{1, 2\} = \{2, 1\}$. This symmetry will be important when it comes time to define rational numbers and ordered pairs.

[3] Or, some would say, without words.

3.4 Set Intersections

Intuitively, the intersection of two sets A and B, denoted $A \cap B$ is the set containing every member which appears in both A and B. To use a not-yet-defined example once again, if A is the set of all even numbers, and B is the set of all prime numbers, then $A \cap B = \{2\}$. Note that the 2 still appears in the curly braces: the intersection is not 2, it is the *set containing* the number 2. Formally defined, we have:

Definition Intersection of sets:

$$(x \in (A \cap B)) \Leftrightarrow ((x \in A) \,\&\, (x \in B))$$

3.5 Set Unions

The next natural relation defined in this lesson is the set union. The union of two sets A and B, denoted $A \cup B$, is the set composed of all mathematical objects which are members of either set A or set B. The object doesn't need to be in both, but it needs to be in at least one.

Definition Union of sets:

$$(x \in (A \cup B)) \Leftrightarrow ((x \in A) \wedge (x \in B))$$

Notice the remarkable similarities between the definitions of the intersection and union of sets. Replacing "and" (&) with "or" ($\wedge$) makes all the difference in the world.

3.6 Set Differences

The final relation defined is the difference between two sets, denoted $A \setminus B$. This is the opposite of the union: $A \setminus B$ is the set containing every member of A which is *not* a member of B. Formally,

Definition Set difference:

$$(x \in (A \setminus B)) \Leftrightarrow ((x \in A) \wedge (x \notin B))$$

Notice once again that there is only a single difference between this and the definition of the intersection of sets.

4 Upcoming Lessons

We will return to set theory and revisit complements and other constructs later in the series. Before returning to purely set theoretic notions, we will use set theory to formally and rigorously define the whole and natural numbers. We will then use it to define addition and multiplication. With the simplest numbers and operations established, the possibilities for application open up considerably.