Math From Scratch Lesson 3: Defining Whole Numbers

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1 What Is A Number?

On its most primal level, mathematics is a system of logic and rules that regulate interactions between strictly defined mathematical objects. The mathematical objects most people are familiar with are numbers. How do we strictly define what a number is?

We begin by defining the set of whole numbers.[1] The only tools we have available to us right now are those of set theory, so that is where we begin.

We do not wish to use our arbitrary historical definitions of numbers to formally define them for all cases. After all, the number "10" is only meaningful in that representation to those using the arabic number system. Even then, the only reason ten is the lowest whole number we express with two digits is that,

[1]The whole numbers, as they are best known, are 0, 1, 2, 3, and so forth. The natural numbers traditionally start at 1 instead of 0. Some set theorists define the natural numbers starting from 0, because it is a more natural definition of numbers, as we are about to see. However, this series will use the more common definition, noting that those who originally named the sets made an error, in that the whole numbers are "more natural" than the natural numbers. This happens in fields which depend on discovery. For example, "atom" means indivisible, and we now know with certainty that they are anything but.

biologically, we happen to have ten fingers.[2] We want a completely independent and formal definition, which can be treated in a fairly absolute sense. To this end, we count.

We can define the number 10 as a set containing 10 objects. Similarly, the number 3 would be the set containing 3 objects. But which objects should we use? There are at least two conventions for that, but both define 0 and 1 the same way.

1.1 Zero and its Successors

We begin by defining zero as a set containing zero members, which is the empty set. So,

$$0 = \emptyset$$

We build the next numbers by defining a mathematical successor. The successor of x is denoted x'. There are two conventions for this definition. One is that $x' = \{x\}$, but that is not the one that will be used in these lessons. (This one defines numbers through subsets of subsets of subsets, much like the nested Matryoshka dolls Russia is known for.) The preferred definition for this project is one which explicitly has a set number of distinct elements as follows:

Definition Successor to x: $x' = (x \cup \{x\})$

With this definition, we can proceed to build our numbers as follows:

$$\begin{aligned}
1 &= 0' = \emptyset \cup \{\emptyset\} = && \{\emptyset\} \\
2 &= 1' = 1 \cup \{1\} = && \{\emptyset, \{\emptyset\}\} \\
3 &= 2' = 2 \cup \{2\} = && \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
4 &= 3' = 3 \cup \{3\} = && \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\} \\
5 &= 4' = 4 \cup \{4\} = && \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}\}
\end{aligned}$$

That's a whole heck of a lot of $\emptyset$s and curly brackets to sift through, but if

[2] In fact, some cultures didn't bother putting the two hands together, and developed number systems based on groupings of five instead.

you do it carefully, you'll see a pattern emerge:

$$\begin{aligned} 1 &= \{0\} \\ 2 &= \{0, 1\} \\ 3 &= \{0, 1, 2\} \\ 4 &= \{0, 1, 2, 3\} \\ 5 &= \{0, 1, 2, 3, 4\} \end{aligned}$$

We can now define the whole numbers as the set $\mathbb{W}$ of all numbers which can be constructed in this fashion, which means 0 and all of its successors. The natural numbers $\mathbb{N}$ can then be defined as 1 and all of its successors, or $\mathbb{N} = \mathbb{W} \setminus 0$.

2 Operations

Mathematical operations are actions we can perform on numbers to change or modify them in some way. The two most common operations are addition and multiplication, which we shall define now.[3]

2.1 Defining addition

Now that we have these things called numbers, it would be nice if we had something we could actually do with them. The simplest operation to define is addition. It is defined by mounting up successors. The successor of x is x', the successor of x' is x'' (which is also the second successor of x), and so forth.

Definition Addition of two whole numbers a and b is defined such that $a + b$ is equivalent to the bth successor of a.

In other words, $x + 1 = x'$, $x + 2 = x' + 1 = x''$, and so forth. Thus, in practice, we can use the set theoretic notions to add as follows (using successor notation but dropping the set notation so I don't have to track several dozen

[3]Subtraction and division will be defined later.

pairs of brackets):

$$\begin{aligned} 7+8 &= 7'+7 = 7''+6 \\ &= 7'''+5 = 7''''+4 \\ &= 7'''''+3 = 7''''''+2 \\ &= 7'''''''+1 = 7'''''''' \\ &= 8''''''' = 9'''''' \\ &= 10''''' = 11'''' \\ &= 12''' = 13'' \\ &= 14' = 15 \end{aligned}$$

Granted, this methodology is far less efficient than those we are used to. The point of this exercise is to prove that we can, indeed, define the whole numbers, natural numbers and addition using methods that are entirely independent of the mode average number of fingers human beings are born with.

2.2 Defining multiplication

Thankfully, a formal definition of multiplication in this methodology is very similar to the one we are used it. We define multiplication as repeated addition.

Definition The multiplication of two whole numbers a and b is defined such that $a \times b = \underbrace{a + a + \ldots + a}_{b \text{ instances of } a}$.

Thus, we have

$$2 \times 3 = 2+2+2 = 2'+1+2 = 2''+2 = 2'''+1 = 2'''' = 3''' = 4'' = 5' = 6$$

We are now armed with the two fundamental and basic operations in algebra. It's time to start putting them to use, starting with our next lesson: defining algebraic groups.