Math From Scratch Lesson 5: Structuring Symmetry - Applications of Group Theory

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1 Introduction To Group Theory

In our last lesson, we learned that groups are formed by the combination of a set of numbers (which we'll now denote G) with an operation $\cdot$ subject to four axioms, or conditions:

1. If $a \in G$ and $b \in G$, then $(a \cdot b) \in G$.
2. If a, b and c are all members of G, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. G contains an element e such that $a \cdot e = e \cdot a = a$ for every possible $a \in G$.
4. Every element $a \in G$ has an inverse $b \in G$ such that $a \cdot b = b \cdot a = e$.

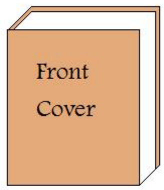

Figure 1: An unrotated book.

Figure 2: A book rotated through rotation x.

Notice that we are *not* required to have $a \cdot b = b \cdot a$ in a group. This is known as the commutative property. In our normal day to day lives, we have never seen an operation in which two numbers don't commute. So, our first step is to show why groups are defined without the commutative axiom. In other words, we need to find a system that cannot be represented by numbers and operations that commute.

2 Rotations and Reflections

Imagine a book with its front cover facing you, as in figure 1. Now imagine a rotation we will label x so that the front cover is turned down, as in figure 2. Add a second rotation y, which takes the original unrotated book and rotates it into the position seen in figure 3, with the spine facing you and the front cover to the right. What happens when we apply these rotations together, instead of one at a time?

If we apply rotation x before rotation y, we arrive at figure 4. If we apply rotation y before x, we arrive at figure 5. These are clearly not the same figure.

This is our first concrete example of variables with an operation that do not commute. Algebraically, $x \cdot y \neq y \cdot x$. When the variables x and y represent rotations, instead of simple numbers, we can no longer guarantee that order is not relevant to the operation involved.

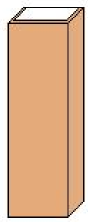

Figure 3: A book rotated through rotation y.

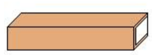

Figure 4: A book rotated through x, then y.

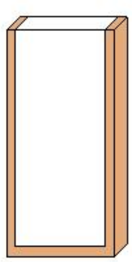

Figure 5: A book rotated through y, then x.

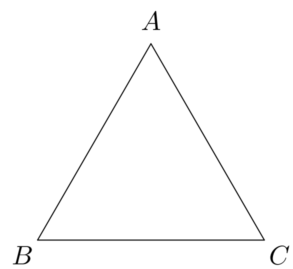

Figure 6: An equilateral triangle.

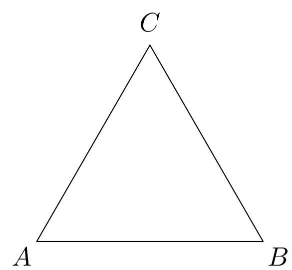

Figure 7: Our earlier equilateral triangle, rotated counterclockwise through a third of a full circle. ($x\Delta$)

3 Example Group: Symmetries of an Equilateral Triangle

Let us examine the ordinary equilateral triangle in figure 6.

This has several symmetries which can be described using group theory. For example, if x represents a counter clockwise turn through $120°$ ($\frac{2\pi}{3}$ radians[1]) then by applying x to our triangle ($x\Delta$) we arrive at figure 7.

If not for our entirely arbitrary human decision to label the vertices A, B and C, the triangles would be identical and indistinguishable. A second rotation by x ($x^2\Delta$) leaves us with figure 8. Again, the symmetries are clear; if not for our labels, the triangles would be indistinguishable.

A third application ($x^3\Delta$) leaves us with our original triangle:

These are rotational symmetries. The symmetries we are more accustomed to studying in school are reflectional symmetries, which can be defined by lines

[1]Readers unfamiliar with radians at this stage can be assured these lessons will eventually define them formally. Radians and degrees are related in a similar manner to metric meters and imperial yards; one can convert one form of angle into another by multiplying through by a set conversion factor.

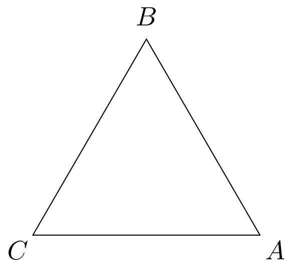

Figure 8: Our earlier equilateral triangle, rotated counterclockwise through two thirds of a full circle. ($x^2\Delta$)

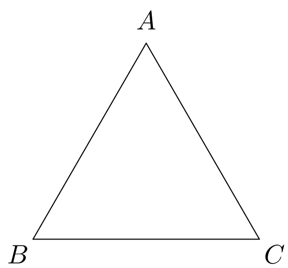

Figure 9: Our earlier equilateral triangle, rotated counterclockwise through a full circle. ($x^3\Delta$)

of symmetry. Our original triangle has three such lines:

Let us use y to represent a reflection about the vertical line (through vertex A.) Thus, $y\Delta$ becomes the triangle seen in figure 11.

We shall now verify that we can create a group with the fundamental building blocks x, y, and the identity element e, which represents doing nothing to the triangle. Our final group will have more than these three members in its algebraic set, but all can be formed using these combinations.[2] Each of the

[2]In fact, we will have six elements in the end. Every combination will end up with the

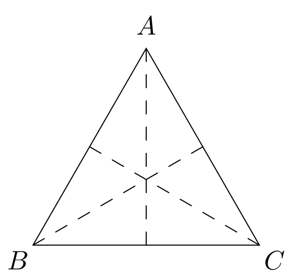

Figure 10: An equilateral triangle, with lines of reflectional symmetry included.

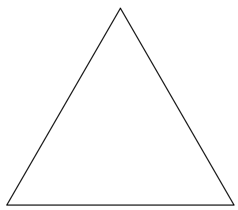

Figure 11: Our equilateral triangle, reflected about the vertical axis of symmetry. ($y\Delta$)

group axioms will now be checked in turn.

3.1 Identity Axiom

Though this wasn't the first axiom on our list, it is often the easiest to check. Yes, it is possible to represent the identity as a symmetry in this group: e may be the rotation through an angle of 0, as measured in either degrees or radians. Thus, the group has an identity.

3.2 Closure Axiom

A "closed" group is one in which every possible product of elements produces another element of the group. We shall now determine whether or not this group is closed with merely x, y and e. Well, it is not. Applying x twice produced figure 8, which does not correspond to any of the other figures we have. Therefore, our group must contain at least e, x, x^2, and y. We also know that y only represented the reflection in the vertical line, so we will also need transformations which correspond to reflections along the other two lines of reflectional symmetry. The obvious combinations to try with this notation are $xy\Delta$ (figure 12) and $x^2y\Delta$ (figure 13), which appear to do the job.[3]

three vertices of our triangle A, B and C permuted in some order, and there are only six permutations possible for those three variables.

[3]Note that the variable closest to the Δ is the one that acts first. The transformations are actually read from right to left; this is the standard in algebra. It "feels" backwards, but only because those raised in English and other Latin-based languages are used to reading from left to right. The notation used for algebra now was developed by Arabic mathematicians, who were used to reading from right to left. Every couple of centuries, teachers in the "western" world try to teach students to add, subtract, multiply and divide multidigit numbers from the left to the right in an attempt to make math "more natural." Every attempt to date has ended in failure when they try to impose this unnatural order on the mathematics the students are trying to cope with.

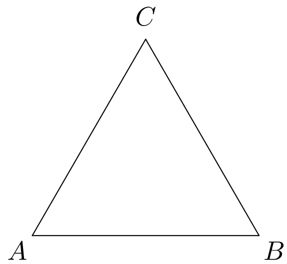

Figure 12: Our equilateral triangle, reflected vertically before being rotated counterclockwise once ($xy\Delta$).

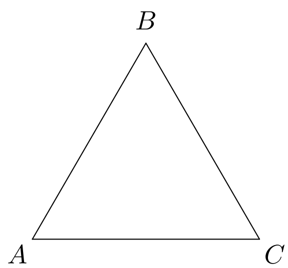

Figure 13: Our equilateral triangle, reflected vertically before being rotated counterclockwise twice ($x^2y\Delta$).

One can now explicitly verify that these six elements (e, x, x^2, y, xy and x^2y) now form a closed group: multiplying any element on either side by any other element produces a result already on the list.

The complete list of results can be represented in table form as seen in table 1. The table is read as follows: to look up the result of the product $(xy)\left(x^2y\right)$, you find the row with xy on the left, the column with x^2y at the top, and then find the product in the space between them. The table can be built in two different ways. One is by explicitly drawing out the transformations in question and comparing them to the table, and the other is somewhat more elegant, covered in detail in subsection **??**.

Note that the Δ we have been using to represent triangle our variables are acting on has been dropped from this notation. It is exceedingly rare to use the Δ at all. In fact, the author has not seen this particular set of mathematical training wheels in any other text, but the author does find it useful to think of things in that manner until one is accustomed to reading the variables from right to left. It is useful to drop the notation early on, as we have done here, as there are applications of group theory which do not lend themselves to such notations easily.

Look closely at the table, and you may see something of particular note:

	e	x	x^2	y	xy	x^2y
e	e	x	x^2	y	xy	x^2y
x	x	x^2	e	xy	x^2y	y
x^2	x^2	e	x	x^2y	y	xy
y	y	x^2y	xy	e	x^2	x
xy	xy	y	x^2y	x	e	x^2
x^2y	x^2y	xy	y	x^2	x	e

Table 1: The table of products for our triangle symmetry group.

each row and each column contains all six elements once each. Every product in this group is unique. This is true of all groups, but it is not true of all algebras. It is a direct consequence of the existence of inverses.[4]

3.3 Associativity Axiom

The associativity axiom is most easily verified using the above table. One can explicitly do so by working out every possible three variable combination in the table. However, that's needlessly time consuming. A somewhat preferred solution is to notice that the only combinations which lead to doubts are those which would have either x or y divided by the other variable. (For example, verifying that $x(xx) = (xx)x$ is unnecessary, as is $x(xy) = (xx)y$, but $x(yx) = (xy)x$ needs to be checked explicitly.) An even more preferred solution is presented in subsection 3.5.

3.4 Existence of Inverses

This is easy to check, thanks to table 1. All we need to do is confirm that there is an e in every row and every column. There is. Note also that the e elements are symmetrically placed about the diagonal from the upper left to the lower right: each element commutes with its own inverse. This is not true of other entries: $yx \neq xy$. This commutativity is a requirement of inverses.

Thus, the fourth and final axiom has been verified. This is a group.

[4] If you have a case in which $ac = bc$, then if c has an inverse, you can multiply by c on the right and find $a = b$. To have $ac = bc$ when a and b are distinct, you cannot be allowed to cancel the c, so it cannot have an inverse.

3.5 Lack of Commutativity of Triangle Symmetries

The elegant way to produce table 1 requires only the explicit drawing of one triangle. We cannot assume that our elements x and y commute in general. We can, however, assume that each element commutes with itself ($xx = xx$ by virtue of indistinguishability.) Thus, we only need to answer one question: what does yx correspond to on our table? Explicitly drawing the triangle leaves us with x^2y. We can now use this to build the elements of our table through the elegance of algebra itself. For example, rather than explicitly drawing multiple triangles, we can determine the value of $(x^2y)(x^2y)$ in the bottom right corner as follows (as we have already verified associativity):

$$\begin{aligned}
(x^2y)(x^2y) &= x^2(yx)xy \\
&= x^2x^2(yx)y \\
&= x^2x^2x^2y^2 \\
&= xxxxxxyy \\
&= (xx^2)(xx^2)y^2 \\
&= e^3 \\
&= e
\end{aligned}$$

By removing the triangles from the procedure, we have not only saved paper from a lot of sketching, but we have managed to reduce the symmetries to their algebraic behaviour, which makes it easier to generalize their application to other situations with the same properties.

4 Upcoming Lessons

The next lesson will elaborate on the other primary application of group theory, which is describing permutations. After that, we will look at subgroups before moving on to the next type of algebra: rings.