Math From Scratch Lesson 8: Mappings

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1 Other Relations

In our very first lesson, we defined relations as "a means of comparing two or more mathematical objects." At that time, the goal was to formally establish the relation of equality, but that relation is far from the only possible relation.

If one takes a careful look at the definition of a relation, one might notice the lack of one particular restriction. There is absolutely no need for the mathematical objects being compared to be the same type of object. Indeed, one could define a relation R such that "xRy is true if and only if $o(x) = o(y)$." Note that this relation compares x and y by their order, as defined in our previous lesson. The order of an element is defined independently of the group it came from; x could be one of the symmetries of an equilateral triangle, while y could be one of the possible permutations on 87 symbols. We will now deal in detail with a specific class of relation.

2 Functions and Mappings

One class of relation that high school students are introduced to is the *function*. The term function is synonymous with a *mapping*, and both terms survive the years because they produce different intuitive mental pictures for the student, each picture having its own benefits to the learning process. The notation of such relations changes somewhat by convention: instead of writing a function relating x and y as yfx in our previous notation, we typically write $y = f(x)$ instead.

Definition A *mapping* or *function* f is a relation between $x \in A$ to $y \in B$ by means of some sort of explicit rule, such that $y = f(x)$ when the rule provided transforms the value x into the value y, subject to the following condition: if $x_1 = x_2$, then $f(x_1) = f(x_2)$. Another notation emphasizing the sets over the elements is $f : A \mapsto B$.

Let us examine this condition, in the context of our modified notation. The typical notation for a function is $y = f(x)$ instead of yfx specifically to emphasize the condition that sets functions apart from other relations. We can treat each x value as an input value, which the function f then transforms into the output value y. The condition "if $x_1 = x_2$, then $f(x_1) = f(x_2)$" can then be thought of as a guarantee that the function will give consistent output when given consistent input.

To emphasize this distinction, think of a television remote control.[1] Selecting the channel you are viewing by use of the number pad is a function; if you input the number 4, your system outputs the audio and video from one particular source, and this source does not depend on which channel you were tuned to before entering the input on the remote control. Every time you enter the number 4, you are tuned to the same channel. However, the "channel up" and "channel down" are not mathematical functions: the result of pressing "channel up" when on channel 3 will send you to channel 4, but repeated use of that button will send you to different channels. Identical inputs lead to non-identical results.[2]

In public school, this condition is typically presented as the [3] or [4]. In that presentation, one is able to graph the ordered pairs (x, y) produced by the

[1]Those with less basic home theatre setups can think of the remote control for the digital tuner, satellite receiver, or other equivalent device which determines which input signal is output to the audio and video devices.

[2]Note: if you only receive one channel in your system, then your "channel up" and "channel down" buttons do behave like functions. However, if you only receive one channel, one must ask why you own a television in the first place.

[3]vertical line rule

[4]vertical line test

function with $y = f(x)$, and then one can draw vertical lines at various places to make sure each vertical line crosses the function no more than once. (They are allowed either one or zero times, but no more.) This is a manifestation of this condition: a vertical line is defined as all points (x, y) which have a particular value for x. If a vertical line crosses the graph of a relation more than once, such as when the relation is the graph of a circle, then we must have two possible y values for a single x, violating the above condition. It may still be a perfectly valid relation between x and y, but it is not a function.

2.1 Differing Terminology

If we have two different terms for the same object, and each has its advantages, when do we use them? The term "mapping" is typically preferred when one wants to emphasize the ability these relations have to relate elements of one set to a completely different set. In other words, it can "map" elements in A onto elements in B. The term "function" is typically preferred when one wants to emphasize the "rule" aspect that accomplishes the transformation from x into y, particularly when $A = B$. In public school, the term "function" is typically used exclusively, as in virtually all cases, $A = B = \mathbb{R}$.[5]

2.2 Example: Order

We have already seen an example of a function, as one may have noticed from the similarity in notation. The notation for order uses the notation of a function or mapping, although the term "mapping" would be preferred in this context. Let x be an element of G, the set which defines the group $(G, \cdot)$. The function $o(x)$ is a mapping from G to $\mathbb{N}$, where $\mathbb{N}$ was defined in lesson 3. G and $\mathbb{N}$ do not need to have anything in common at all to define this mapping.

3 Properties of Functions

Once functions have been introduced, we find that there are many accompanying definitions and properties that are key to discussing them. These will be defined here.

[5] The symbol $\mathbb{R}$ will be defined formally in a later lesson, probably in the spring of 2012.

3.1 Domain

The *domain* of a function is the set of all possible elements $x \in A$ such that $y = f(x)$ is defined. In many cases, the domain of f is the entirety of A, but this is not true in all cases.

For example, let A be the set of geometric symmetries of a square. Imagine that $y = f(x)$ is the function that maps a rotation x into the angle y through which the shape rotates.[6] In this case, the domain of $f(x)$ is limited only to those elements x which represent rotations; reflections are not rotations, so there is no meaningful quantity for $f(x)$ to map the result to. The reflections are a part of the set A, but they are not in the domain of $f(x)$. Note that the identity would be in the domain of $f(x)$: it corresponds to a rotation through angle 0 in whatever units you choose to measure your angle with.

3.2 Range

The *range* of a function $y = f(x)$ is the set of all $y \in B$ which the function can map values onto. As with the domain, this may not be the entire set. For example,[7] let us examine the order function. Let A be the set of all permutations on four symbols. If our function $f(x) = o(x)$, then $B = \mathbb{N}$. If we examine all possible permutations on four symbols, we find that none of them have order higher than four. Thus, the entire range of the order function using this particular domain is $\{1, 2, 3, 4\}$. This does not come remotely close to using the entire infinite set of $\mathbb{N}$. Note that several different points in A can map to the same point in B: there is exactly one element with order 1 in any set (the identity), but this set has nine elements[8] of order 2, eight[9] of order 3 and six[10] of order 4.

Note that, when one wishes to refer to B and not simply the range of f, one can use the term *codomain* to do so.

[6] Again, please forgive the author for using examples derived from operations and mathematical definitions not yet defined. We simply have not yet established enough concrete mathematical objects to create clear examples any other way. Note, though, that such "cheating" is used for examples only, and not for the foundation of the theory. Once we have the axioms of a field established, likely in early 2012, this will no longer occur. Angles will eventually be defined.

[7] For once, this example is defined entirely using mathematical objects we have already introduced.

[8] $(12), (13), (14), (23), (24), (34), (12)(34), (13)(24)$ and $(14)(23)$

[9] $(123), (132), (134), (143), (124), (142), (234)$ and (243)

[10] $(1234), (1243), (1324), (1342), (1423)$ and (1432)

3.3 Injections

Some functions are called *injective functions* or *one-to-one functions.* These are functions with one special property.

Definition Let f be a function which maps domain D into set B, such that (using the set notation introduced in lesson two) we find that the range of the function R is a subset of set B, or $R \subseteq B$. f is an *injective* or *one-to-one* function if there are no two distinct points in the domain which map to the same point in the range. Formally, $\forall a, b \in D, f(a) = f(b) \Leftrightarrow a = b$.

For example, if the domain of f is the set of whole numbers $\mathbb{W}$, and the function itself is defined as $f(x) = x^2$, then f is an injective function.[11] Note that the range is only a subject of the whole numbers $\mathbb{W}$. For example, although the number 2 is in the domain, it is not in the range.

Taking another common example, the order function is *not* an injection. Using the permutations on four symbols as the domain, we have nine different elements in the domain which map to the element 2 in the range, eight which map to 3, and six which map to 4.

3.4 Surjections

Definition In direct contrast to an injective function, a *surjective function* or an *onto function* is a function which maps set A into set B such that the range $R = B$. In other words, for any possible $y \in B$ there is at least one x (and quite possibly more) such that $f(x) = y$. Formally: let $f : A \mapsto B$. $\forall y \in B, \exists x \in A$ such that $f(x) = y$.

3.5 Bijections

Definition A *bijective function* f creates a *one-to-one correspondence* (not to be confused with a one-to-one function) between two sets A and B if the function $f : A \mapsto B$ is both injective and surjective. In other words, every point in the domain maps to a different point in the range, and every point in the range is mapped to from some point in the domain.

[11] This would *not* be injective if the domain was the set of integers, but we haven't formally defined integers yet. As the set of whole numbers contains only positive numbers and zero, there are no two points in the domain which map to the same point in the range.

A basic example: if the set A is the basis of a group $(A, \cdot)$, then the function $f : A \mapsto A$ defined as $f(x) = x^{-1}$ is a bijection. Every point has an inverse, and we have already proven (in lesson four) that no two points share the same inverse.

In our previous lesson, we discussed the *cardinality* of a group, as the formal means to discuss the size of a group. When we explore this in more detail later, bijections will be of vital importance. It is not a difficult intuitive leap to see that two sets related through a bijective function must have the same number of elements, and therefore be the same size. The interesting points which can be teased but not proven at this point: a set A may contain every element of set B and more, and the two sets may still have the same size; infinity comes in infinitely many dramatically different sizes; the permitted sizes of infinity are discrete: if the smallest infinity has the size $\aleph_0$, then the second smallest infinity has the size $\aleph_1 = 2^{\aleph_0}$, the third smallest infinity has the size $\aleph_2 = 2^{\aleph_1} = 2^{2^{\aleph_0}}$, and so forth.

4 Upcoming Lessons

With mappings and functions defined, we are now equipped to define cosets and move on to Lagrange's theorem, one of the most important results of group theory.