Math From Scratch Lesson 13:
Metric Spaces Other Than Canada

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1 Preliminary Work: Functions With Multiple Rules

There are times in which it is convenient to define a function with one rule for part of its domain and a completely different rule for another part of its domain. The notation for this is as follows:

$$f(x) = \begin{cases} \text{Rule 1}, & \text{Conditions in which we use Rule 1} \\ \text{Rule 2}, & \text{Conditions in which we use Rule 2} \\ \vdots & \\ \text{Rule } n, & \text{Conditions in which we use Rule } n \end{cases}$$

The simplest example is perhaps the step function $u(x)$:

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

This function has the value of 0 if a number is negative, or 1 if a number is nonnegative.[1]

2 Metric Spaces

In lesson 10, we established the axiom of inequality, which allowed us to show that integers (as well as whole and natural numbers) could be sorted in a particular order. What we have not yet established is how far apart they are. We know that they come in the order of the successors, as defined in lesson two. In other words, the number that immediately follows a is $a + 1$. Is the distance between a and $a + 1$ well defined? If so, is it the same as the distance between $a + 1$ and $a + 2$? We can explore this by defining the metric function.

Definition A function $d(x, y)$ which maps set S onto set R is a *metric function* if it satisfies the following conditions:

1. x and y are members of the same set S.
2. $d(x, y)$ maps onto a set R which is subject to the axiom of inequality. Note that the set S does *not* need to conform to the axiom: only R does.
3. $d(x, y) \geq 0 \ \forall x, y \in S$
4. $d(x, y) = 0$ if and only if $x = y$
5. $d(x, y) = d(y, x)$
6. $d(x, z) \leq d(x, y) + d(y, z) \ \forall x, y, z \in S$

A *metric space* (S, d) is formed by the combination of a set S and a metric function d which satisfies the conditions of a metric function on the set S.

Note that condition 6 has no particular requirements on *which* elements are chosen as x, y and z. This is known as the *Triangle Inequality*, for reasons that will be made clear when we can define triangles.[2]

[1] Note the use of "nonnegative" instead of "positive" in the terminology. 0 is neither positive nor negative, so "positive" would leave the function undefined at 0, while "nonnegative" does not have this limitation.

[2] We're getting closer! All we need to do is to use metric functions to define the number line, define vector spaces, define vector spaces and number lines in two dimensions, define division, use division to establish the rational numbers, define convergent series, use infinite convergent series to define the real numbers, define ordered coordinate pairs, define straight line segments connecting those ordered pairs in a vector space, establish a metric function applicable to such two dimensional spaces, and then determine how to graph such things. Once we've done all of that, defining triangles is trivial.

Definition The value of the metric function $d(x,y)$ is the *distance* between points x and y in our metric space (S,d).

Note the flexibility of the definition. The answer to both of our previous questions[3] is the same: "it is if we want it to be." We could easily define a metric function $\mathbb{Z}$ as

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

This satisfies every condition above, but makes comparing distances virtually impossible: $d(1,2) = d(3,10) = d(-105,2000000) = 1$. A more intuitive definition would be

$$d(x,y) = x - y$$

There is a problem with this, however. This could easily violate condition 3 if $y > x$, not to mention its complete inability to conform to condition 5 unless $x = y$. We have a couple of options for dealing with this. In lesson 10, we were able to define positive and negative numbers in a rigorous sense. We noted that a negative number, when multiplied by itself, is a positive number. Moreover, $-a \cdot -b = a \cdot b$. Thus, we would define our distance metric as

$$d(x,y) = (x - y)^2$$

This would satisfy conditions 3 and 5 easily. However, it would violate condition 6: let $x = 0$, $z = 3$ and $y = 1$ and check it for yourself. It is also convenient if $d(n, n+m) = d(r, r+m) = d(m, 0)$ were true for all possible values of m, n and r, as it would mean that the distances between points was consistent. There is a way to define such a metric function, if we can first define the absolute value function.

2.1 Absolute Value

We shall now define a mapping from a set S onto itself such that, for every $x \in S$

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In other words, if x is zero or a positive number, nothing changes. If x is a negative number, then it becomes its additive inverse, which is a *positive*

[3] Is the distance between a and $a+1$ well defined? If so, is it the same as the distance between $a+1$ and $a+2$?

number. This function is useful and common enough to get its own name, and it is called the *absolute value* function. Note that R must be the basis of an algebra which is subject to the axiom of inequality for this function to apply as defined here. Later on, when we study number systems which are *not* subject to the axiom of inequality,[4] then we will introduce a new definition of absolute value which is more generally applicable.

Not only does the absolute value function get its own name, it gets its own notation: $f(x) = |x|$. It also has some quick and useful implications: $|x| = |-x|$ is one, and the other is that, whenever $|x| = |y|$, it must also be true that either $x = y$ or $x = -y$. We can write this more compactly as $x = \pm y$.

Now we can define a convenient metric function on $\mathbb{Z}$:

$$d(x, y) = |x - y|$$

Let us now show that this is, indeed, a metric space. The conditions for a metric space are as follows:

A function $d(x, y)$ which maps set S onto set R is a *metric function* if it satisfies the following conditions:

1. x and y are members of the same set S.

   Proof This part is trivial: we can define this as such. In this case, $S = \mathbb{Z}$.

2. $d(x, y)$ maps onto a set R which is subject to the axiom of inequality.

   Proof Again, we simply choose $R = \mathbb{Z}$. As subtraction is merely adding the negative of a number, we know that $x - y\mathbb{Z}$ because the integers are closed under this operation. As for the absolute value, $|x| \in \mathbb{Z}$ is trivially obvious for $x \geq 0$. If $x < 0$, then we note that any time $x \in \mathbb{Z}$ it is also true that $-x \in \mathbb{Z}$, and we are done. We can impose a further demand: as $d(x, y) \geq 0$, we can choose $R = \mathbb{W}$ instead.

3. $d(x, y) \geq 0 \ \forall x, y \in S$

   Proof The definition of the absolute value ensures this is true.

4. $d(x, y) = 0$ if and only if $x = y$

   Proof If $|x - y| = 0$, then $x - y = x + (-y) = 0$. Thus, $y = x$ by our definition of subtraction.

[4] The complex numbers are notable here.

5. $d(x, y) = d(y, x)$

   Proof This amounts to proving that $|x - y| = |y - x|$. Well, $x - y = x + (-y) = 1 \cdot (x + (-y)) = (-1)^2 \cdot (x + (-y)) = (-1) \cdot (-x + y) = -(y - x)$. By the definition of the absolute value, $|y - x| = |-(x - y)| = |x - y|$.

6. $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in S$

   Proof This is the trickiest one. The instinct is to draw a number line, label the points, and argue based on that. However, we cannot use that method, as we need to prove that $d(x, y) = |x - y|$ is a metric function before we can justify drawing a number line in the first place!

   Let us pick three arbitrary integers, and label them such that $x \geq y \geq z$. With this, we know the following:

   (a) $|x - y| = x - y \geq 0$
   (b) $|x - z| = x - z \geq 0$
   (c) $|y - x| = x - y \geq 0$
   (d) $|y - z| = y - z \geq 0$
   (e) $|z - x| = x - z \geq 0$
   (f) $|z - y| = y - z \geq 0$

   We now need to examine the three cases that follow:

   (a) $d(x, z) \leq d(x, y) + d(y, z)$
   (b) $d(x, y) \leq d(x, z) + d(y, z)$
   (c) $d(y, z) \leq d(x, y) + d(x, z)$

   We do this by recalling our definition of the $\leq$ relation: $d(x, z) \leq d(x, y) + d(y, z)$ is true when $d(x, y) + d(y, z) - d(x, z) \geq 0$.

   (a) Case 1: $d(x, z) \leq d(x, y) + d(y, z)$

   $$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= \\ |x - y| + |y - z| - |x - z| &= x - y + y - z - x + z = 0 \geq 0 \end{aligned}$$

   (b) Case 2: $d(x, y) \leq d(x, z) + d(y, z)$

   $$\begin{aligned} d(x,z) + d(z,y) - d(x,y) &= \\ |x - z| + |z - y| - |x - y| &= x - z + y - z - x + z = 2(y - z) \geq 0 \end{aligned}$$

(c) Case 3: $d(y,z) \leq d(x,y) + d(x,z)$

$$\begin{aligned} d(x,y) + d(x,z) - d(y,z) &= \\ |x-y| + |x-z| - |y-z| &= x - y + x - z - y + z = 2(x-z) \geq 0 \end{aligned}$$

That proves each possible case. Keep in mind that, if you have three variables already labelled x, y and z which do not conform to $x \geq y \geq z$, then you can simply relabel them as a, b and c in such a way that $a \geq b \geq c$ is true, and then repeat the proof that way. In other words, assuming that $x \geq y \geq z$ has not, in any way, impacted the validity of this logic.

With this result now proven, we can check to see if the remarkably convenient properties we hoped for earlier will hold:

1. Does $d(a, a+1) = d(a+1, a+2)$?
2. Does $d(n, n+m) = d(r, r+m) = d(m, 0)$ hold for all possible n, m and r?

Notice that we only have to check the second condition. If $n = a$, $r = a+1$ and $m = 1$, then the second condition becomes the first. This is actually quite easy to confirm:

$$\begin{aligned} d(n, n+m) = &\quad |n-n-m| &= |-m| \\ d(r, r+m) = &\quad |r-r-m| &= |-m| \\ d(m, 0) = &\quad |m-0| = |m| &= |-m| \end{aligned}$$

With all of this established, we can define a number line.

3 Number Line

One of the first concepts introduced in school is the number line. We have finally reached the point where we can justify the visual representation of numbers in this fashion. The axiom of inequality allows us to put the integers in a single, consistent, and most importantly linear order, while the distance between numbers defined through the metric $d(x,y) = |x-y|$ allows us to choose to graph the numbers on a line with a consistent distance between the points. The number line depicted in figure 1 is a familiar sight to most of us.

Notice some of the important features: both ends of the line end with arrows. This is the convention amongst mathematicians to signify that the line continues

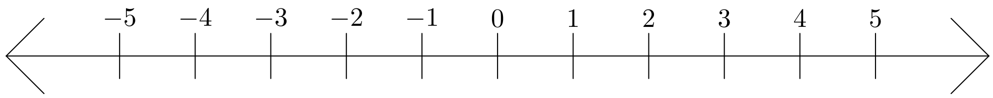

Figure 1: The number line.

for an infinite distance off the edge of the page. Note also that it is drawn as a horizontal line, rather than vertical. This is the most common convention for a line used to represent integers, but as we'll later see, this is far from a requirement. Note also that the positive numbers are written to the right of the zero, and that the negative numbers are written to the left. This is also a human convention rather than a mathematical requirement. Note also that the negative numbers are listed in "reverse" order, relative to the positive numbers. That *is* a mathematical requirement, given that, if $a > b$, then $-b > -a$, as established in lesson ten. Notice that there is a mirror symmetry in the placement of the integers and their negatives, as well. This is required by our choice of metric: $d(m, 0) = |m| = d(-m, 0)$, meaning that any arbitrary integer m is just as far from 0 as its negative. This one is *not* a requirement of the math; we could define another metric which does not conform to this property. However, if we use a different metric, we must draw the number line differently. Notice also the vertical lines which indicate each number's placement on the line. This convention is very common with a horizontal number line. However, it can be confusing when we reach vector spaces in multiple dimensions, requiring both horizontal and vertical number lines. Typically, once that stage is reached, notations change. We will deal with this when the time comes. For now, just note which elements are required by the mathematics of the situation and which are simply conventions humans have chosen to adopt.

4 Upcoming Lessons

We have finally equipped ourselves with all of the tools needed to define division in a useful fashion. This will be the focus of the next lesson.