Math From Scratch Lesson 16: Divisibility Rules

W. Blaine Dowler

May 30, 2011

Contents

1 Divisibility and Remainders

When most of us were first introduced to division, it was in a limited context. We were taught that four was divisible by two, but that five was not. In that first introduction, we were effectively taught that one number could only be divided by another if the remainder was zero. While there are definitely interesting results that require zero remainders, they are certainly not the only option.

The division algorithm is a means to find the solutions to equations of the form

$$d = aq + r$$

where $0 \leq r < |d|$, and d and a are given. This process can get somewhat lengthy, particularly if one is interested in determining whether or not the number 1345984621 is divisible by 7. Using remainders, results about common factors and our knowledge of base 10 mathematics, we will be able to create rules to determine whether or not a number is divisible evenly by another number[1] $2 \leq a \leq 11$ with fewer steps than the complete division in most cases.

2 The Concept

The concept of the divisibility rule is borne of the basis representation theorem. For example, the number 1234 is actually a shorthand for $1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10 + 4$. We shall break the number 10 into multiples of our divisor a to collect terms that will not contribute to the remainder of the full division. We will transform our large dividend d into a smaller number that will be easier to divide, eventually turning any arbitrarily large finite number into a much smaller number s. While this number won't always be positive, we can ensure that $0 \leq |s| \leq |a| - 1$.

3 Division by 2

By the basis representation theorem, any finite dividend d can be written in the form

$$d = d_0 + d_1 \cdot 10 + d_2 \cdot 10^2 + \ldots + d_m \cdot 10^m$$

for some finite m. This can be rewritten in the form

$$d = d_0 + \left(d_1 + d_2 \cdot 10 + \ldots + d_m \cdot 10^{m-1}\right) \cdot 10$$

[1] We start with 2 because the divisibility by a is the same as that for $-a$, because nothing is divisible by 0, and because all integers are divisible by 1.

by factoring a single instance of 10 out of the latter terms in the sum. Now, we are looking to determine the value of r an equation of the form

$$d = 2q + r$$

We can do this by equating our two expressions for d, and noting that $10 = 2 \cdot 5$.

$$2q + r = d_0 + \left(d_1 + d_2 \cdot 10 + \ldots + d_m \cdot 10^{m-1}\right) \cdot 2 \cdot 5$$

If our *only* interest is in knowing the value of r, then we can eliminate every term known to contain a factor of 2. After all, by definition, $0 \leq r \leq 2 - 1 = 1$, meaning r is either 0 or 1. By our convention with the basis representation theorem, the terms involving 2 will not contribute to the final value of r. Thus, all contributions to r must come from d_0, the digit in the ones place of d. We are now reduced to finding the remainder of dividing a one digit number by 2, which is much more manageable than d itself if, say, m were 60.

As 2 happens to be a factor of 10, we get an additional simplification. We can determine the divisibility by inspection. If $d_0 \in \{0, 2, 4, 6, 8\}$, then the remainder is guaranteed to be 0. If, however, $d_0 \in \{1, 3, 5, 7, 9\}$ then the remainder will be 1. This is equivalent to the grade school version of "if the ones digit is even, then the number is divisible by 2."

4 Divisibility by 3

This rule is a bit different from the rest. We set up the process with $10 = 3 \cdot 3 + 1$. We then note that $10^2 = (3 \cdot 3 + 1) \cdot (3 \cdot 3 + 1) = 81 + 18 + 1 = 33 \cdot 3 + 1$. By induction,[2] one can show that, in fact, $10^n = z \cdot 3 + 1$ for some integer z.

With this, we write

$$d = d_0 + d_1 \cdot 10 + d_2 \cdot 10^2 + \ldots + d_m \cdot 10^m$$

as

$$d = d_0 + d_1 + d_2 + \ldots + d_m + 3 \cdot z$$

for some z, which depends entirely on the individual values of d_i. However, as this z is being multiplied by 3, it does not contribute to the remainder r in

$$3q + r = d_0 + d_1 + d_2 + \ldots + d_m + 3 \cdot z$$

[2] In the next lesson, we introduce the Chinese Remainder Theorem which demonstrates this much more effectively, which is why the proof is left out for now. Induction serves well enough for this application-driven lesson.

Thus, we find that our remainder r is the same remainder as we'll get from dividing the sum $d_0 + d_1 + d_2 + \ldots + d_m$ by 3. Thus, the remainder when dividing d by 3 is the same as the remainder obtained when dividing the sum of the digits of d by 3. This same rule applies to the result, too. So, if dividing 1231654987621 by 3, we can take $1+2+3+1+6+5+4+9+8+7+6+2+1 = 55$ and apply the rule again to find $5+5 = 10$. A third application gives us $1+0 = 1$, so that the remainder when dividing $1231654987621 \div 3 = 1$. We can always apply the rule repeatedly until $s = d_0 + d_1 + d_2 + \ldots + d_m$ is subject to the constraint $0 \leq s \leq 9$. At this stage, there are no digits to add together. Now, if $0 \leq s \leq 2$, then $r = s$. if $3 \leq s \leq 5$, then $r = s - 3$, and if $6 \leq s \leq 8$, then $r = s - 6$. If $s = 9$, then $r = 0$.

5 Divisibility by 4

There are two options for checking divisibility by 4. The first option is that typically covered in public school, created by writing

$$d = d_0 + d_1 \cdot 10 + d_2 \cdot 10^2 + \ldots + d_m \cdot 10^m$$

as

$$d = d_0 + d_1 \cdot 10 + \left(d_2 + \ldots + d_m \cdot 10^{m-2}\right) \cdot 10^2$$

Since $10^2 = 100 = 4 \cdot 25$, we see that the series in brackets will not contribute to the remainder when a number is divided by 4. Thus, we need only concern ourselves with the portion of d that appears in the tens and ones places, or $d_0 + d_1 \cdot 10$. The remainder when dividing this two digit number by four will be the same as when we divide the entire number by four. However, one can take this a step further and reduce our test number far more effectively by noting that $10 = \cdot 2 \cdot 4 + 2$, so that we further refine our test number to $d_0 + d_1 \cdot 2$. Thus, we can test the number 12234234109824876 for divisibility by 4 either by testing 76 or by testing $7 \cdot 2 + 6 = 20 = 2 \cdot 2 + 0 = 4$ instead. The latter method can be applied repeatedly to ensure that our final result s is a number less than 10. Thus, we have several cases to test again: if $0 \leq s \leq 3$, then $r = s$. If $4 \leq s \leq 7$, then $r = s - 4$, and if $8 \leq s \leq 9$ then $r = s - 8$.

6 Divisibility by 5

This is much like divisibility by 2, as 5 is the other factor of 10. Using identical logic, we can quickly show that the remainder r of $d \div 5$ will be the same as the remainder when dividing $d_0 \div 5$. The primary difference is that we have more than two possible remainders, so the inspection step takes a little more thought.

If $d_0 \leq 4$, then $r = d_0$. If $5 \leq d_0 \leq 9$, then $r = d_0 - 5$. Thus, not only can we quickly determine that 125123428 is not divisible by 5 as we could with the public school rule, we can also quickly state that the remainder when divided by 5 is $8 - 5 = 3$.

7 Divisibility by 6

As with 4, there are two possible tests here, corresponding to the one that was taught in public school and another one that is now available. The one taught in public school is usually that one should apply the tests for 2 and 3. As $6 = 2 \cdot 3$, if 6 is divisible by 2 and 3, then any number divisible by both of those will also be divisible by 6. While this is certainly true, the author has seen is provided without proof, caveat, or explanation, resulting in problems when students then test divisibility by 8 by checking for divisibility by 2 and 4 separately, which gives a "false positive" result as often as it accurately predicts divisibility by 8. The reason that this test works for 6 is that the factors 2 and 3 are relatively prime. This test does *not* work when the two factors themselves have common factors, as with the factors 2 and 4 which divide into 8.

An alternative test is available which eliminates this possibility. We begin by noting $10 = 6+4$. Now we note that $(6 \cdot n + 4) \cdot 10 = 6 \cdot m + 40 = 6 \cdot m + 36 + 4 = 6 \cdot l + 4$ for some integer values of n, m and l.[3] Thus, we can rewrite

$$d = d_0 + d_1 \cdot 10 + d_2 \cdot 10^2 + \ldots + d_m \cdot 10^m$$

as

$$d = d_0 + (d_1 + d_2 + \ldots + d_m) \cdot 4 + (\ldots)$$

and compute the number

$$s = d_0 + (d_1 + d_2 + \ldots + d_m) \cdot 4$$

to test our remainder. In short, we multiply each digit save the ones digit by 4, and then add the results to the ones digit. (For efficiency in calculation, it is probably best to add all but the ones digit first, add then up, *then* multiply by four, and *then* add the ones digit.) Again, this process can be continued until our test number s is in the $0 \leq s \leq 9$ range. With this, we find our test cases once more: if $0 \leq s \leq 5$, then $r = s$. Otherwise, $6 \leq s \leq 9$ and $r = s - 6$.

[3] When we are only concerned with the remainder when divided by 6, the specific values of n, m and l are irrelevant.

8 Divisibility by 7

There is a divisibility test by 7, although it is rarely taught in public school. This is likely because the rule is cumbersome, and is not that much easier to calculate than outright division. This is because the various powers of 10 do not have simple and easy remainders when divided by 7. In fact,

$$\begin{aligned}
10 &= 7 \cdot 1 + 3 \\
100 &= 7 \cdot 14 + 2 \\
1000 &= 7 \cdot 142 + 6 \\
10000 &= 7 \cdot 1428 + 4 \\
100000 &= 7 \cdot 14285 + 5 \\
1000000 &= 7 \cdot 142857 + 1 \\
&\vdots
\end{aligned}$$

This produces a cyclic process, such that

$$s = d_0 + 3 \cdot d_1 + 2 \cdot d_2 + 6 \cdot d_3 + 4 \cdot d_4 + 5 \cdot d_5 + d_6 + 3 \cdot d_7 + \ldots$$

Once again, repeated application is guaranteed to eventually result in a one digit number. If $0 \leq s \leq 6$, then $r = s$. If $7 \leq s \leq 9$, then $r = s - 7$. However, this is hardly an easy rule to remember, which would need to be looked up or rederived with virtually every application. Even the author finds outright division to be more efficient in this particular case.

9 Divisibility by 8

The case for divisibility by 8 is much like that for 4. Public schools teach the simple "if the last three digits are divisible" rule based on the fact that $1000 = 8 \cdot 125$, thereby creating a test that is simple to apply, but which may result in a test number s as high as 999. One can note that $10 = 8 + 2$ and $100 = 96 + 4 = 8 \cdot 12 + 4$ to further simplify our test case into

$$s = d_0 + 2 \cdot d_1 + 4 \cdot d_2$$

Repeated application once again brings this down to the $0 \leq s \leq 9$ range, with a simple set of test cases once more. When $0 \leq s \leq 7$, then $r = s$, but when $8 \leq s \leq 9$ then $r = s - 8$.

10 Divisibility by 9

This divisibility test is very much like that for 3. As $10 = 9 + 1$, and as $10^n \cdot 10 = 10^n \cdot 9 + 10^n$, we quickly find that our test number s is given by

$$s = d_0 + d_1 + d_2 + \ldots + d_m$$

Repeated application will, once more, reduce s to a single digit number with $r = s$ for $0 \le s \le 8$ and $r = 0$ for $s = 9$.

11 Divisibility by 10

This is, perhaps, the most obvious divisibility rule in the set with this methodology. It is straightforward to see that $s = d_0$, with no additional calculations required, leaving us with $r = s$.

12 Divisibility by 11

The final divisibility rule explicitly derived here is that for divisibility by 11. It is notable that $10 = 11 - 1$, and that $100 = 9 \cdot 11 + 1$. Proof by induction may be used to show that 10^n will always be one less than a multiple of 11 for odd n, and will always be one more than a multiple of 11 for even n. Thus, we find that our test number s is given by

$$s = d_0 - d_1 + d_2 - d_3 + \ldots \pm d_m$$

where the final sign before d_m depends upon whether m is odd or even. For the first time, we *cannot* guarantee that we will end with $0 \le s \le 9$ in the repeated and unthinking application of this process! For example, the number 10 gives $s = -1$. We must check our resulting s with each step to see if the value is positive or negative. If our *only* concern is for identifying the case $r = 0$, with no regard for non-zero remainders, then we may simply ignore the negative sign and repeat the process. In the case that the remainder is important, and if our $s < 0$, then we can reapply the process, treating each d_i as the negative number that it is. This is equivalent to reversing the addition and subtraction signs in our above expression for s. In this manner, we can eventually reach a value for s such that $0 \le s \le 10$, and then $r = s$.

13 Higher Divisibility Tests

Tests for divisibility by numbers above 12 are entirely possible and can be derived in much the same manner as that for divisibility by 11. However, these rules will not be covered here.

14 Divisibility in Other Bases

If one finds oneself working in bases other than base 10 on a regular basis, it may be worth deriving comparable rules for that base. They do need to be rederived from scratch, as they change dramatically. (For example, in base 6, the rule for divisibility by 3 looks much like the divisibility by 5 rule in base 10. In turn, the divisibility by 5 rule in base 6 looks much like the divisibility by 9 rule in base 10.)

15 Upcoming Lessons

We next establish the Chinese Remainder Theorem, apply it specifically to some computer applications (including random number generation and cryptography) and then move on to algebraic fields and the rational numbers.