# Math From Scratch Lesson 25: Infinite Processes

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## 1 Processes

A mathematical *process* is a sequence of steps which result in a specific result. For example, addition is a process for finding the sum of two numbers. Using the set theoretic notation from lessons 2-4 means addition is a process involving a large number of simple steps:

$$\begin{aligned} 6+7 &= 6+6' = 6+5'' = 6+4''' = 6+3'''' = 6+2''''' = 6+1'''''' \\ &= 6''''''' = 7'''''' = 8''''' = 9'''' = 10''' = 11'' = 12' = 13 \end{aligned}$$

Using the later notation, addition becomes a small number of more complex steps:

$$6+7=13$$

Either way it is a process to obtain a specific result. Now that we are armed with the field of rational numbers, we can do a lot more than that. For example, we can define a recursive process.

## 1.1 Recursive Processes

A *recursive* process is one in which each step refers to the step before. For example, let us examine the process defined as follows:

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) \\
x_3 &= \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right) \\
x_4 &= \frac{1}{2}\left(x_3 + \frac{2}{x_3}\right) \\
&\vdots \\
x_{n+1} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)
\end{aligned}$$

Each step requires the results of the previous step. The results of the first few steps can be found in table 1. Note that this process seems pretty arbitrary at this point, but we will return to it later. It will reveal itself to be rather significant in the future, as the process will eventually demonstrate that there are numbers on the number line which are *not* rational numbers.

Now that we have established our process, we have to ask ourselves: how long can this process continue? Each step of the process has been labeled and counted with a natural number, shown by the subscript on each number. There is no particular reason to assume this process needs to stop before we run out of natural numbers. So, how many natural numbers are there?

| Iteration | $x_n$ |
|---|---|
| $x_1$ | 1 |
| $x_2$ | $\frac{3}{2}$ |
| $x_3$ | $\frac{17}{12}$ |
| $x_4$ | $\frac{577}{408}$ |
| $x_5$ | $\frac{665857}{470832}$ |

Table 1: The results of our recursive process with $n = 4$.

# 2 Infinity

Numbers are, to a large degree, theoretical objects. This is true of sets as well, including the sets that numbers are made out of. For any given natural number $x$, there is a successor $x' = \{x, \{x\}\} = x + 1$ which is greater than $x$. Each natural number is less than its successor, and every natural number has a successor.

This leads to a result that early cultures often struggled with: there is no highest number.

There are *infinitely many* numbers available. We represent the "number" known as *infinity* with the symbol $\infty$. It is important to note that infinity is not actually a number. It is something beyond numbers, which cannot be represented by any set or combination of sets. Although we have no way to prove it yet, it shouldn't even be referred to in the singular; there is more than one distinct infinity. (In fact, there are infinitely many infinities.) There is a specific infinity worth mentioning though, and that is the infinity that represents the number of natural numbers: that is the smallest infinity, $\aleph_0$. We will also prove that $\aleph_0$ is what is known as a *transfinite* number, meaning it is its own successor. In short, $\aleph_0 = \aleph_0 + 1$. Why do we not have $\aleph_1 = \aleph_0 + 1$? Because the different sizes of infinity have a very specific relationship to each other. There are rules about how big relative infinities can be[1] which dictate that the next smallest infinity after $\aleph_0$ is $\aleph_1 = 2^{\aleph_0}$.

It is exploring the concept of infinity that opens the doors to the first truly fascinating and counter intuitive results in math. Several of these results will be coming in the next few lessons.

# 3 There Is More To Life Than Rationality

Let us examine our above process. If we follow it for one step, the result is 1. If we follow it for two steps, the result is $\frac{3}{2}$. If we follow it for five steps, the result is $\frac{665857}{470832}$, and so forth. Each of these steps is a rational number, since it is a combination of rational numbers, which is a closed algebra under the addition and multiplication operations we are using here, right?

Our process can be considered to be complete should we ever reach a point where

$$x_{n+1} = x_n$$

[1] We will, eventually, manage to get to these rules, but it won't be any time soon.

or

$$x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

Under what conditions does this take place? Let us apply our rules of algebra to determine what value $x_n$ must take for this condition to be satisfied.

$$\begin{aligned}
x_n &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\
x_n &= \frac{1}{2}\left(\frac{x_n^2 + 2}{x_n}\right) \\
x_n &= \frac{x_n^2 + 2}{2x_n} \\
2x_n^2 &= x_n^2 + 2 \\
x_n^2 &= 2
\end{aligned}$$

Thus, the process ends if and when we reach a stage where $x_n^2 = 2$. When dealing with the integers, we saw that there was no solution to this: it would violate the fundamental theorem of arithmetic, as prime factors on the left of the $=$ sign appear with even exponents while those on the right feature only $2^1$. We are no longer restricted to the integers, though. We can seek out a solution within the set of rational numbers.

A number is a rational if it can be written in the form

$$x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}$$

So, if we have $x_n^2 = 2$ with $x_n = \frac{p}{q}$ then we must also have

$$\frac{p^2}{q^2} = 2$$

We have also seen that we can factor both numbers and their inverses. So, if the numbers have any common factors, we can cancel those out, ensuring $p$ and $q$ are relatively prime.[2] For example, if we start with $x = \frac{6}{8}$ we can write

$$x = \frac{6}{8} = \frac{2 \cdot 3}{2 \cdot 4} = 3 \cdot 2 \cdot 2^{-1} \cdot 4^{-1} = 3 \cdot 1 \cdot 4^{-1} = \frac{3}{4}$$

where 3 and 4 are relatively prime, or coprime.

[2] In elementary school, we called this a “reduced” fraction.

Let us examine

$$x_n^2 = \frac{p^2}{q^2} = 2$$

and see what we can make of it. This is equivalent to

$$p^2 = 2q^2$$

which transforms both sides of the equation into integers.

Let us now apply the fundamental theorem of arithmetic. We know the right hand side has a factor of two. Therefore, in any possible solution, the left hand side must also have a factor of two. Therefore, $p$ has a factor of 2, so that $p = 2k$ for some appropriate $k$. Thus,

$$2^2k^2 = 2q^2$$

or

$$2k^2 = q^2$$

since 2 is a cancelable element. By the same logic, as the left hand side now has a factor of 2, we must have $q = 2m$ for some $m$. This reduces the problem to solving

$$k^2 = 2m^2$$

This is identical to the above problem; the restrictions on $k$ and $m$ are identical to those on $p$ and $q$! Our solution must be an infinite process! Moreover, we have actually found a contradiction. We initially assumed that $p$ and $q$ had no common factors. Well, we have now shown that both $p$ and $q$ must have factors of 2 in order to find a rational number such that

$$x_n^2 = \frac{p^2}{q^2} = 2$$

This leaves us with a single, inescapable conclusion: there is no rational number $x_n$ which satisfies this. Taking it a step further results in one of two obvious conclusions: either there are numbers that are not described by the set of rational numbers, or the recursive process defined above never ends. As it turns out, both are true.

There is also a third conclusion that is far more subtle. The rational numbers are closed according to one of the axioms we used to define them. This axiom, as well as some of the others, applies only when we deal with *finite* combinations of rational numbers. We will soon learn that we need to add several[3] numbers to our number line in order to come up with an algebraic field which retains the closure property even after adding infinitely many numbers to the process. Unfortunately, some of our other axioms cannot withstand the weight of the infinite process even then.

[3]By "several" I mean "infinitely many." In fact, although we will not be able to prove it for some time, we need to add not only $\aleph_0$ infinitely many, but $\aleph_1$ infinitely many.

## 4 Next Lesson

In the next lesson, we establish the notions of sequences and series in a rudimentary form.