Math From Scratch Lesson 26: Finite Sequences and Series

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Contents

1 Sequences

In mathematics, a *sequence* is a list of numbers that follow some sort of rule or pattern. The rules can be extremely arbitrary, to the point where the rule might be assigning every number in the sequence a value. The list of numbers must be countable and in a particular order. For example, one cannot define a sequence as "the elements of the set $\{3, 1, 2\}$ because that set is no different than the set $\{1, 2, 3\}$, the set $\{3, 2, 1\}$, and so forth. We label each *term* in a sequence with a number, typically written as a subscript to the variable t (for term). For example, if we make a sequence of the first n natural numbers starting with 1 and continuing in order, we have

- $t_1 = 1$
- $t_2 = 2$
- $t_3 = 3$

- $t_4 = 4$
- $t_5 = 5$
- $t_6 = 6$
- $t_7 = 7$
- $\ldots$
- $t_n = n$

The rules are typically defined through some form of algebraic equation. The two most common types of sequences are *arithmetic* and *geometric* sequences.

1.1 Arithmetic Sequences

An arithmetic sequence is one in which the "rule" for determining the value of each term is given by the equation

$$t_n = a + (n - 1) \cdot d$$

where t_n is the nth term in the sequence, a is the first term in the sequence ($a = t_1$) and d is known as the "common difference." These sequences are usually introduced as "number patterns" in the earliest years of mathematical awareness, as they are equivalent to saying "count by ds, starting with a." So, when a young student counts by 2s as "2, 4, 6, 8, 10 . . .", that student is listing the sequence provided by letting $a = d = 2$.

An arithmetic sequence is defined by the fact that it can be defined recursively. It fits the pattern $t_{n+1} = t_n + d$ regardless of n. Similarly, $t_m = t_n + d \cdot (m - n)$.

1.2 Geometric Sequences

A geometric sequence is comparable to the arithmetic sequence, save for determining each subsequent term through multiplication by a constant r rather than addition of a constant d. A sequence is geometric if it can be written in the form

$$t_n = a \cdot r^{n-1}$$

where a, n and t_n have the same definitions as above. Similarly, the terms can be defined recursively as $t_{n+1} = r \cdot t_n$, or $\frac{t_{n+1}}{t_n} = r$, which is why r is referred to as the common ratio. We also have $\frac{t_m}{t_n} = r^{m-n}$.

2 Series

When treated in isolation, sequences have limited applications. Sequences are primarily useful as a precursor to series. If you have a sequence of terms t_n, then the series based on this is the sequence

- $S_1 = t_1$
- $S_2 = t_1 + t_2$
- $S_3 = t_1 + t_2 + t_3$
- $\ldots$
- $S_N = t_1 + t_2 + t_3 + \ldots + t_N$

where each term S_n is the sum of the first n terms of the sequence.

With the standard definitions of arithmetic and geometric sequences, we can also find that there are standard expressions for each series, and these expressions can be derived and expressed in more than one way.

2.1 Arithmetic Series

An arithmetic series is of the form

$$S_N = t_1 + t_2 + t_3 + \ldots + t_N = a + a + d + a + 2d + \ldots + a + (N-1) \cdot d$$

Logic indicates that this will result in the equation

$$S_N = N \cdot a + d \cdot (1 + 2 + 3 + \ldots + (N-1))$$

This can simplify further if we can develop an expression for $1+2+3+\ldots+(N-1)$. We can do this through either pure logic, or proof by induction.

2.1.1 $1 + 2 + 3 + \ldots + N$ By Logic

As N is a natural number, it will be either even or odd. At first, assume N is even. We can rearrange our sum such that

$$1+2+3+\ldots+N = (1+N)+(2+N-1)+(3+N-2)+\ldots+\left(\frac{N}{2}+\frac{N}{2}+1\right)$$

If one looks carefully at the brackets on the right hand side, one will notice that every bracket contains a set of numbers whose sum is $N+1$. As there are $\frac{N}{2}$ such sets of brackets (i.e. $\frac{N}{2}$ pairs of numbers, pairing the first with the last, the second with the second last, etc.) the final expression is

$$1+2+3+\ldots+N = N \cdot \frac{N+1}{2}$$

Now we examine the case when N is odd. We can construct a similar rearrangement, but now the final number has no pairing, so that we have

$$1+2+3+\ldots+N = (1+N)+(2+N-1)+(3+N-2)+\ldots+\frac{N+1}{2}$$

In this case, we have $\frac{N-1}{2}$ pairs of numbers which add up to $N+1$, and an additional $\frac{N+1}{2}$ which has no pairing.

The sum of these terms is given by

$$\frac{N-1}{2} \cdot (N+1) + \frac{N+1}{2} = (N+1) \cdot \left(\frac{N-1}{2} + \frac{1}{2}\right) = N \cdot \frac{N+1}{2}$$

which is identical to the expression we had for the case when N is odd. Thus,

$$1+2+3+\ldots+N = N \cdot \frac{N+1}{2}$$

regardless of whether N is odd or even.

It should be noted that there is a flaw in this proof. It depends on the ability to rearrange the terms above under the commutative property of addition without changing the sum. The commutative property only applies to finite sums; when we cover infinite sequences and series in a forthcoming lesson, this proof will not be valid.

2.1.2 $1+2+3+\ldots+N$ By Induction

An inductive proof, first introduced in volume one of these lessons, requires a certain amount of either intuition or perspiration. One must already have the equation, and then must simply use algebra to prove that it is, indeed, true. We show that the equation is true for $N=1$, and then prove that if the equation is true for the case $N=n$, then it must also be true for the case $N=n+1$.

To show that it is true for $N=1$, we substitute it in. The sum $1+2+3+\ldots+N=1$ when $N=1$. Our expression

$$N \cdot \frac{N+1}{2} = 1 \cdot \frac{1+1}{2} = \frac{2}{2} = 1$$

is identical, as $1 = 1$. Therefore, this works in the trivial case. Next, we must prove that truth in the case of $N = n$ implies truth in the case of $N = n + 1$.

In the case of $N = n + 1$, we have

$$\begin{aligned} 1+2+3+\ldots+n+n+1 &= \frac{n\,(n+1)}{2}+n+1 = \frac{n\,(n+1)+2n+2}{2} = \frac{n^2+3n+2}{2} \\ &= \frac{(n+1)\,(n+2)}{2} = \frac{(n+1)\,((n+1)+1)}{2} \end{aligned}$$

which is exactly what we were looking for. Thus, we have proven the equation inductively, without depending upon rearranging the terms.

With these in place, we can now state the value of an arithmetic series as

$$S_N = N \cdot a + d \cdot \frac{N\,(N-1)}{2} = N \cdot (a + d \cdot (N-1))$$

2.2 Geometric Series

With a geometric series, our initial expression is somewhat different, but the process is still the same. In this case, the sum can be expressed as

$$S_N = a \cdot \left(1 + r + r^2 + r^3 + \ldots + r^{N-1}\right)$$

and it is the $1 + r + r^2 + r^3 + \ldots + r^{N-1}$ portion that needs to be simplified. This requires a little more effort on our parts to assemble.

We begin by multiplying the sum by $1 - r$ for no apparent reason. [1] This produces

$$\begin{aligned} & (1-r) \cdot \left(1 + r + r^2 + r^3 + \ldots + r^{N-1}\right) \\ = \; & 1 + r + r^2 + r^3 + \ldots + r^{N-1} - r - r^2 - r^3 - \ldots - r^{N-1} - r^N \\ = \; & 1 + r - r + r^2 - r^2 + r^3 - r^3 + \ldots + r^{N-1} - r^{N-1} - r^N \\ = \; & 1 - r^N \end{aligned}$$

which means that

$$(1-r) \cdot \left(1 + r + r^2 + r^3 + \ldots + r^{N-1}\right) = 1 - r^N$$

or

$$1 + r + r^2 + r^3 + \ldots + r^{N-1} = \frac{1 - r^N}{1 - r}$$

[1] The real reason to do it is because experience tells us this can work, where "experience" is defined as something you obtain just after you need it.

provided $r \neq 1$. In the case $r = 1$, every term is identical, so the sum is simply N.

Again, this logic is only valid for a finite sum, as it depends upon applying both the distributive and commutative properties to the sum itself. An inductive proof is possible, but is not particularly constructive or instructional at this stage.

3 Next Lesson

In our next lesson, we will start examining infinite series, and show how such series can violate the closure, associative and commutative properties of both addition and subtraction, and prove that the rational numbers alone are insufficient for describing every number on the number line.