Math From Scratch Lesson 27: Infinite Sequences and Series

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September 25, 2012

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1 Infinite Sequences

Infinite sequences are not particularly different from finite sequences. The primary difference is in the definition: instead of defining the sequence as running from $n = 1$ to $n = N$, we merely state that the index $n \in \mathbb{N}$. More generally, one could even define the sequence with $n \in \mathbb{Z}$, extending it to infinite length in both directions. Both arithmetic and geometric sequences adapt extremely well to this notion.

With this flexibility established, we can now generalize our finite sequences to cover any range of indices: we merely take a subset of the infinite sequence that is covered by $n \in \mathbb{Z}$. Things start to get hairy when we move into infinite series.

2 Infinite Series

An infinite series is a much trickier thing than an infinite sequence. The difference is subtle but fundamental, and best demonstrated with quantitative examples of infinite series.

We can start with reasonable examples. For example, let us take the standard geometric sequence

$$t_n = ar^{n-1}$$

with the parameters $a = 0$ and $r = 2$. This sequence is $\{0, 0, 0, 0, 0, 0, \ldots\}$. For any finite N, the sum of the series is

$$S_N = \frac{a\left(1 - r^N\right)}{1 - r} = \frac{0\left(1 - 2^N\right)}{1 - 2} = 0$$

so it seems reasonable to assume that any infinite series based on this sequence would also have a sum of 0. Note that this is not a proof that this exists, but it is a pretty compelling argument that this would work.

Let us now take as an example the geometric sequence with parameters $a = 1$ and $r = -1$. This sequence becomes $\{1, -1, 1, -1, 1, -1, 1, -1, \ldots\}$. The finite series over N terms has two possible terms. If N is even, then $S_N = 0$, but if N is odd, then $S_N = 1$. This makes for a problem with an infinite series: which of the two possible sums is correct? Is neither one correct? Can both somehow be correct?

Let us look at a third example series, taking the geometric sequence $a = 1$, $r = 2$ and examining what that becomes. This sequence is then $\{1, 2, 4, 8, 16, 32, 64, \ldots\}$. We can use the basis representation theorem to show that this is equivalent to the binary number $\underbrace{11111111\ldots1111}_{N\ digits}$. Again, working in binary numbers, we find that

$$S_N = 1 + 2 + 4 + 8 + \ldots + 2^{N-1} = 2^N - 1$$

for any finite N. If we try to extend this to an infinitely large N, we run into a problem: how do we define an infinite exponent? This doesn't fit any definition of exponents we have seen thus far. Why do sequences work so well and series work so poorly? The difference is subtle:

- An infinite *sequence* is an infinite collection of finite calculations. Each calculation is based on the axioms of algebra, which are defined with (at most) three variables and then extended through a finite process to the rest.
- An infinite *series*, assuming we can successfully define one at all, requires infinitely many *infinite* calculations. We must first prove that the infinite series can even be defined.

This is the challenging part of defining the real numbers. The rational numbers alone, in the absence of infinite series and other infinite calculations, form a perfectly valid algebraic system. Much of what follows will be stated without proof at this stage. This is not intended to be the final proof; this will come later. Our goal here is to lay a mental roadmap for the surprises coming up so that one is prepared for the counterintuitive results. We will eventually be able to show that infinite series are perfectly valid, but some with unexpected considerations.

3 Validity of the Axioms

Let us review the axioms of algebra we used to define the rational numbers and mark the ones which are questionable in infinite processes.

3.1 Closure under +

This axiom will fail with infinite processes. A concrete example can be drawn from material that we are not even close to covering at this stage: the irrational number e. Readers unfamiliar with this number need only know two things:

1. e is irrational
2. e is the sum of the infinite series $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

As $n!$ is an integer for all integers n, we know that $\frac{1}{n!}$ is a rational number for all integers n. Thus, this infinite sequence of rational numbers does *not* have a rational sum.

3.2 Closure under $\cdot$

This axiom will fail with infinite processes. If we use the symbol Π for infinite products, we can write

$$\prod_{n=1}^{\infty} \frac{4 \cdot n^2}{4 \cdot n^2 - 1} = \frac{4}{3} \cdot \frac{16}{15} \cdot 3635 \ldots = \frac{\pi}{2}$$

where π is the usual irrational number.

3.3 Commutativity under +

This axiom also fails. Let us look to our previous example of

$$\sum_{n=1}^{\infty} = (-1)^{n-1} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \ldots$$

If commutativity holds, then we can put "all" of the +1 terms first and "all" of the −1 terms afterwards. This is a problem, since we never run out of +1 terms. Instead of a sum alternating between 1 and 0, we have a series in which $S_N = N$ for any finite N, and never reaches the −1 terms to cancel them out. This example had problems before, but the fact that we knew there were only two possible sums and we can now get a completely different option indicates that there is a problem.

3.4 Commutativity under $\cdot$

This also fails. The structure of addition and multiplication are so similar that we will consistently find that the pairs of axioms fail for both operations or neither in most cases. For this one, we simply take our previous example and replace the sum with the product:

$$\prod_{n=1}^{\infty} = (-1)^{n-1} = (1) \cdot (-1) \cdot (1) \cdot (-1) \cdot (1) \cdot \ldots$$

The same rearrangement of terms is the difference between having −1 as the result regardless of N or alternating between 1 and −1.

3.5 Associativity under +

This one is a bit more subtle, but uses the same sequence as the past few examples. If we apply the associativity of addition to

$$\sum_{n=1}^{\infty} = (-1)^{n-1} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \ldots$$

as

$$\sum_{n=1}^{\infty} = (-1)^{n-1} = (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots$$

then we end up with

$$0 + 0 + 0 + 0 + \ldots = 0$$

but we we associate the terms as

$$\sum_{n=1}^{\infty} = (-1)^{n-1} = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots$$

we end up with

$$1 + 0 + 0 + 0 + 0 + \ldots = 1$$

which is a completely different result. Thus, the associative property (which states that rearranging brackets doesn't change the final result) does not apply.

3.6 Associativity under ·

Here we use our (seemingly stock) example with a different bracket arrangement. Starting from

$$\prod_{n=1}^{\infty} = (-1)^{n-1} = (1) \cdot (-1) \cdot (1) \cdot (-1) \cdot (1) \cdot (-1) \ldots$$

we can bracket like so:

$$\prod_{n=1}^{\infty} = (-1)^{n-1} = ((1) \cdot (-1)) \cdot ((1) \cdot (-1)) \cdot ((1) \cdot (-1)) \cdot \ldots$$

and transform our product into $(-1) \cdot (-1) \cdot (-1) \cdot (-1) \cdot (-1) \cdot \ldots$ which alternates with each term, producing and inconsistent series, or bracket the product like so:

$$\prod_{n=1}^{\infty} = (-1)^{n-1} = (1) \cdot ((-1) \cdot (1) \cdot (-1)) \cdot (1) \cdot ((-1) \cdot (1) \cdot (-1)) \ldots$$

taking grouping the factors as either 1 or $((-1) \cdot (1) \cdot (-1))$ producing the product $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \ldots$ which is always 1.

3.7 Identity under +, specifically 0

This axiom holds: $x+0+0+0+0+\ldots = x$ is still true. Similarly, $0+\sum a_n = \sum a_n$ if $\sum a_n$ makes sense at all.

3.8 Identity under ·, specifically 1

This axiom also holds.

3.9 Inverses under + (negatives)

This axiom holds as well. If $\sum a_n$ is well defined, then $(\sum a_n) - (\sum a_n) = 0$.

3.10 Inverses under · (reciprocals)

This was the pair of axioms that had a different structure under finite sequences, since 0 has no reciprocal but every rational number has a negative. In the case of infinite processes, both axioms hold. If $\prod a_n$ is defined and equal to some number $L \neq 0$, then $\frac{1}{\prod a_n} = \frac{1}{L}$.

3.11 Distributive property

The distributive property is the one that gets very strange. Take three infinite sums, $\sum a_n$, $\sum b_n$ and $\sum c_n$. The following two applications of the distributive property are true in all cases:

$$\begin{aligned}\left(\sum a_n\right) \cdot (b+c) &= \left(\sum a_n\right) \cdot b + \left(\sum a_n\right) \cdot c \\ a \cdot \left(\sum b_n + \sum c_n\right) &= a \cdot \sum b_n + a \cdot \sum c_n\end{aligned}$$

The strangeness starts when you multiply one infinite sum by another. In some respects, it is the order of operations that fails, more so than the distributive property. Let $\sum a_n = A$ and $\sum b_n = B$. If you are calculating

$$\left(\sum a_n\right) \cdot \left(\sum b_n\right)$$

by computing the sums in the brackets first and then multiplying them together, you will get the result $A \cdot B$ as one would intuitively expect. If, instead, you multiply one infinite sum through another using the distributive property first as

$$\begin{aligned}\left(\sum a_n\right)\cdot\left(\sum b_n\right) &= a_0\cdot(b_0+b_1+b_2+\ldots)+a_1\cdot(b_0+b_1+b_2+\ldots)\\ &\quad +a_2\cdot(b_0+b_1+b_2+\ldots)+\ldots\\ &= a_0\cdot b_0+a_0\cdot b_1+a_1\cdot b_0+a_2\cdot b_0+a_1\cdot b_1+a_0\cdot b_2+\ldots\\ &= \sum_{k=0}^{n}\sum_{i=0}^{k}a_i b_{k-i}\end{aligned}$$

and define

$$c_k=\sum_{i=0}^{k}a_i b_{k-i}$$

then we have turned our two infinite series into a single infinite series $\sum c_n$. The strange part is this: generally speaking,

$$\left(\sum a_n\right)\cdot\left(\sum b_n\right)\neq\sum c_n$$

When computed individually and multiplied together, the result is as expected. When two series are combined into a single series, the result changes. There are conditions under which the product is as expected, but we aren't ready to go through those at the moment. We will deal with that in greater detail later.

3.12 Axiom of inequality

The axiom of inequality holds in multiple forms. If we have $a_n < b_n \forall n$, then we also have $\sum a_n < \sum b_n$ (assuming such sums are meaningful in the first place.) In the case of infinite products, things get dicey. We can have $a_n < b_n \forall n$ with the definitions

$$a_n = -\left(2^n\right)$$

and

$$b_n = \frac{1}{n}$$

As every a_n is negative and every b_n is positive, the sums of the two sequences preserve the inequality. The products, however, do not:

$$\prod_{n=1}^{N} b_n = \frac{1}{N!}$$

will steadily approach 0 as the product is calculated. Conversely,

$$\prod_{n=1}^{N} a_n = (-1)^N\cdot 2^{\frac{N(N+1)}{2}}$$

will produce results of alternating signs and increasing magnitudes. There is a form of the axiom which does hold, however, when dealing with infinite products: if $|a_n| < |b_n| \forall n$, then $|\prod a_n| < |\prod b_n| \forall n$ any time the infinite products are meaningful to begin with.

4 Summary

What does this all mean? Essentially, we have a choice: we can work exclusively with the 12 axioms we have and with the rational numbers, rejecting infinite sums and products, *or* we can extend our axioms and collections of numbers to include the infinite processes. In the next few lessons, we will be looking at other such choices. Note that we are never *required* to extend the field of rational numbers to do useful math. We only need to do so in order to accommodate some specific applications. One of these applications is calculus, so we'll be choosing to extend our axioms and our set of numbers. The good news is that we can incorporate infinite sequences, infinite series, infinite products and some other mathematical features we want with a single axiom.