Math From Scratch Lesson 28: Rational Exponents

W. Blaine Dowler

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1 Exponent Review

Let us begin by reviewing exponents and what they mean to us so far. We have defined exponents through repeated multiplications, as follows:

$$x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{n\ times}$$

With this definition, we have a few standard properties of exponents we can utilize.

1.1 $x^n \cdot x^m$

We can see that

$$\begin{aligned} x^n \cdot x^m &= \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{n\ times} \cdot \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{m\ times} \\ &= \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{n+m\ times} \\ x^n \cdot x^m &= x^{n+m} \end{aligned}$$

Thus, exponents are additive when we multiply two terms with the same base.

1.2 $\frac{x^n}{x^m}$

This is similar to the previous example, using division instead of multiplication:

$$\begin{aligned} \frac{x^n}{x^m} &= \frac{\overbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}^{n\ times}}{\underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{m\ times}} \\ \frac{x^n}{x^m} &= x^{n-m} \end{aligned}$$

since we can cancel m instances of x in the numerator.

This leads to two questions. What if $n = m$? In this case, we get perfect cancelations from both the numerator and the denominator, resulting in 1 for an answer. Using this formula, we have $x^0 = 1$ because of these perfect cancelations, almost regardless of the value of x.[1]

Our second question: what if $m > n$? Then x^{n-m} results in a negative exponent. This is fine, provided we understand how to interpret those negative exponents. For example:

$$\frac{x^2}{x^3} = \frac{x \cdot x}{x \cdot x \cdot x} = \frac{\not{x} \cdot \not{x}}{\not{x} \cdot \not{x} \cdot x} = \frac{1}{x} = x^{-1}$$

Thus, a negative exponent for x means that factor needs to appear in a denominator. (If the factor of x with a negative exponent is already in the

[1] I say "almost" because $x \neq 0$ as it appears in the denominator. $\frac{0}{0}$ is undefined.

denominator, then it can be moved to the numerator with a positive exponent.) In general,

$$x^{-m} = \frac{1}{x^m}$$

and

$$\frac{1}{x^{-m}} = x^m$$

With this, we extend our definition of exponents beyond the natural numbers and into the full set of integers.

1.3 $(x^n)^m$

Similarly,

$$\begin{aligned}
(x^n)^m &= \underbrace{\underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{n\ times} \ldots \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{n\ times}}_{m\ times} \\
(x^n)^m &= x^{nm}
\end{aligned}$$

Thus, when raising a base with an exponent to an exponent, the exponents multiply.

2 Further Extensions: Rational Exponents

One might ask, now that we have extended exponents from the natural numbers to the integers, if it would not be possible to extend them further into the entire rational set of numbers. It is an excellent question, with a nontrivial answer. First, what would we mean by $x^{\frac{1}{n}}$ to begin with?

For this, we appeal to the above property of exponents: $(x^n)^m = x^{nm}$. If we let $m = \frac{1}{n}$ we arrive at

$$(x^n)^m = (x^n)^{\frac{1}{n}} = x^{\frac{n}{n}} = x^1 = x$$

Thus, somehow, fractional exponents should "cancel" an exponent whose numerator matches the denominator of the fractional exponent.

For example,

$$9^{\frac{1}{2}} = \left(3^2\right)^{\frac{1}{2}} = 3^{\frac{2}{2}} = 3^1 = 3$$

which makes perfect sense: rational exponents allow us to reverse, or undo a previous exponent. The concept was developed far earlier than this notation, which is why we also see notation like $\sqrt{x} = x^{\frac{1}{2}}$, or more generally, $\sqrt[n]{x} = x^{\frac{1}{n}}$. Note the small n in the $\sqrt[n]{\ }$ "radical" notation: it is assumed to be 2 when omitted. When $n = 2$, it is called a "square root," when $n = 3$ it is a "cube root" and when it is higher we simply call it the nth root (e.g. fourth root, fifth root, etc.)

This begs a further question: what happens if we take the square root of something and arrive at an answer which doesn't have an integral exponent? For example, $2^{\frac{1}{2}}$?

3 Exploring Square Roots

3.1 Counting Square Roots

Let us assume that $r^2 = a$, indicating that $r = \sqrt{a}$. We need to determine if this is a unique value or not. We will show that 0 has a unique square root and that all other numbers have *two* square roots.

We know that, of $x \cdot y = 0$, then either $x = 0$ or $y = 0$. If $x = \sqrt{0}$, then $x^2 = x \cdot x = 0$, and thus $x = 0$. Therefore, 0 is its own square root.[2]

Let us assume that $a > 0$. Let us also assume $x^2 = a$ and $y^2 = a$. We now prove that there are only two possibilities for x and y: either $x = y$, or $x = -y$. We prove this by contradiction: assume $x > y$. By the axiom of inequality, if $x > y$, then $a = x^2 > xy > y^2 = a$, which is a contradiction if x and y are both positive: a would need to be strictly greater, and *not* equal to itself. If $x > 0$ and $y < 0$, then the part that fails is the $xy > y^2$ step: the left hand side is negative, while the right hand side is positive. To repeat the logic allowing for negatives, we must instead demand that $|x| \geq |y|$, and our transformed statement becomes $a = x^2 \geq |xy| \geq y^2 = a$. This statement may be valid if and only if we drop the inequalities and $x^2 = |xy| = |x|\,|y| = y^2$. Demanding either $x^2 = |x|\,|y|$ or $|x|\,|y| = y^2$ means $|x| = |y|$, so that either $x = y$ or $x = -y$. This is, of course, based on the assumption that $x > y$; we can repeat the statements for $y > x$ and arrive at the same conclusion.

[2] If $x = x^2$, then x is said to be *idempotent*. We will deal with idempotents in more detail later. For now, suffice it to say that 0 and 1 are the only idempotents in an algebraic field.

3.2 Calculating Square Roots

We can now calculate the positive square root of a positive number, provided we are allowed one assumption: infinite processes are valid. We have already seen that they have questionable validity under the rational number system, but, as the next few lessons will demonstrate, an algebra that allows the existence of infinite processes allows for significantly many more tools in our mathematical toolkit.

Let us again begin by searching for the number $r > 0$ which satisfies $r^2 = a$ when $a > 0$. Let us begin with the first approximation $x_1 \approx r$, and develop a means to refine this approximation. If $x_1 < r$, then $(x_1)^2 < r^2 = a$ by the axiom of inequality. Thus, we also have $x_1 \cdot r < r^2 = a$. If we can find a y_1 such that $x_1 \cdot y_1 = a$, then we will know that $x_1 < r < y_1$. Similarly, if $x_1 > r$, then we would need to find a y_1 such that $x_1 \cdot y_1 = a$ and $y_1 < r < x_1$. We can find the corresponding y_1 the same way in both cases: if

$$x_1 \cdot y_1 = a$$

then

$$y_1 = \frac{a}{x_1}$$

for all $x_1 > 0$. If our first approximation is exactly correct, then $x_1 = y_1 = r$. If not, then either $x_1 < r < y_1$ or $y_1 < r < x_1$. Thus, a better approximation of the square root would be a number between both x_1 and y_1. If $x_1 > y_1$, then we can use the axiom of inequality to show that

$$x_1 = \frac{1}{2}(x_1 + x_1) > \frac{1}{2}(x_1 + y_1) > \frac{1}{2}(y_1 + y_1) = y_1$$

In the case where $x_1 < y_1$, we have instead

$$x_1 = \frac{1}{2}(x_1 + x_1) < \frac{1}{2}(x_1 + y_1) < \frac{1}{2}(y_1 + y_1) = y_1$$

In either case,

$$\frac{1}{2}(x_1 + y_1) = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right)$$

is between x_1 and y_1, just as r is, and so we define our next approximation for r as

$$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right)$$

Generally speaking, we can define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

for n iterations of the process. We now ask ourselves: will this process end? Will we arrive at an exact answer, and if not, will we at least get "closer" to the correct answer?

If x_1 is an exact answer, then x_n will be the same exact answer every iteration from this point on, as

$$x_{n+1} = \frac{1}{2}\left(r + \frac{a}{r}\right) = \frac{1}{2}(r + r) = r$$

regardless of n. Proving that we get "closer" to the correct answer cannot be done at this stage. We will soon show that we cannot guarantee that r is a rational number, and the proof that this does get "closer" to the correct answer each iteration depends on theorems involving more than the rational numbers. Thus, at this stage, the best we can formally say is that we cannot guarantee the validity of this process at this time. We can ultimately show that

$$\frac{|x_{n+1} - y_{n+1}|}{|x_n - y_n|} = \left|\frac{x_n^2 - a}{2\left(x_n^2 + a\right)}\right| < 1$$

proving that the interval represented by $[x_n, y_n]$ gets dramatically smaller with each iteration, and that the interval, in fact, gets arbitrarily small as n gets arbitrarily large. This proof requires elements of calculus and real numbers, which we haven't justified yet.

3.3 Testing Rationality of $\sqrt{x}$

The final check we need to do is to see if $\sqrt{x}$ is rational for all $x > 0$. We will eventually examine the case when $x < 0$ as well, but that is not going to happen any time soon.

Let us choose the specific example of $\sqrt{2}$. If $\sqrt{2} \in \mathbb{Q}$, then we have $\sqrt{2} = \frac{p}{q}$ for some pair of integers p and q. If $\sqrt{2}$ is rational, then we know it can be positive or negative, and we choose to work exclusively with the positive root at this stage, thus ensuring that p and q are both natural numbers. By the definition of the rational numbers, we can write $\frac{p}{q}$ in its lowest terms, meaning that p and q are relatively prime. We now show that assuming $\sqrt{2} \in \mathbb{Q}$ leads to a contradiction.

If $\sqrt{2} = \frac{p}{q} \in \mathbb{Q}$, then $2 = \frac{p^2}{q^2}$, or $2q^2 = p^2$. By the Fundamental Theorem of Algebra, we can show that the left hand side is divisible by 2, and thus the right hand side must also be divisible by 2. In that case, $p = 2k$ for some natural number k, since p^2 cannot be divisible by 2 unless p itself is divisible by 2. We can use this to write $2q^2 = p^2 = 4k^2$, or $2q^2 = 4k^2$, or $q^2 = 2k^2$. Now the right hand side is divisible by 2, so we can argue that $q = 2n$ for some n. This leads to our contradiction: if $q = 2n$ and $p = 2k$, then $\frac{p}{q}$ could have been written as $\frac{k}{n}$, and thus was not originally in lowest terms. We can continue this process infinitely many times should we so choose, but as $\frac{p}{q}$ needs to be finite to be a valid representation of $\sqrt{2}$, we are forced to conclude that there is no such valid representation of $\sqrt{2}$, and $\sqrt{2}$ is not a rational number.

4 Conclusions

If we continue to restrict ourselves to the rational numbers, we will be unable to define fractional exponents for every possible base, although we can do so for certain specific cases. In future lessons, we will find other situations in which we need to move beyond the rational numbers to add new tools to our mathematical toolkit. In lesson 32, we will arrive at the extended system, known as the real number system, and we will get there by adding a single axiom to our set of 12.