Math From Scratch Lesson 29: Decimal Representation

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1 Introducing Decimals

We now have the tools needed to formally introduce decimal numbers. We have seen infinite processes, negative exponents and the bases representation theorem, which are the three tools we need.

We first introduced the bases representation theorem in lesson 11. The core idea is that we can choose a base b, typically choosing $b = 10$, and represent any number as an expansion with powers of this base. For example, the number 243 is shorthand for $2 \cdot 10^2 + 4 \cdot 10^1 + 3 \cdot 10^0$. Specifically, we said the following:

Theorem 1.1 *The bases representation theorem states that, for any given natural number a, there is a unique representation to base k such that*

$$a = a_m k^m + a_{m-1} k^{m-1} + \ldots + a_2 k^2 + a_1 k^1 + a_0$$

such that every $a_i \in \mathbb{W}$, every $a_i < k$, either $a_i = 0$ or $a_i > 0$, and at least one $a_i \neq 0$.

Rewriting this with our series notation, we can state

Theorem 1.2 *The bases representation theorem states that, for any given natural number a, there is a unique representation to base b such that*

$$a = \sum_{i=0}^{m} a_i b^i$$

such that every $a_i \in \mathbb{W}$, every $a_i < b$, either $a_i = 0$ or $a_i > 0$, and at least one $a_i \neq 0$.

We will now extend this idea to represent numbers other than integers through the use of negative exponents and infinite processes by writing the representation as

$$a = \sum_{i=-\infty}^{+\infty} a_i b^i$$

subject to the following:

1. $a_i \in \mathbb{Z} \forall a_i$
2. If $a > 0$, then $a_i \geq 0 \forall a_i$. If $a < 0$, then $a_i \leq 0 \forall a_i$.
3. If $a \neq 0$, then at least one $a_i \neq 0$.
4. $a_i = 0 \forall a_i$ if and only if $a = 0$.

The question now becomes a question of whether or not decimal representations are unique. The answer is "almost." There is one representation that is not unique. We will discover this as we develop an algorithm to convert our rational numbers into decimal representation.

2 Finite Decimals

2.1 $\frac{1}{10}$

Let us begin with one of the simplest rational numbers, namely $\frac{1}{10}$. How would we write this as a decimal? Well, this is the same as $1 \cdot 10^{-1}$, so the representation is given by choosing $a_{-1} = 1, a_i = 0 \forall i \neq -1$. This gives us the infinite series, but it does not yet give us the representation. We cannot simply write down a number that has infinitely many digits as $\ldots 0000001000000 \ldots$ as we did before, as the "place value" of each digit is ambiguous. We use the symbol "." (which we call a decimal point) to distinguish between the digits with $i \geq 0$ and $i < 0$ in our expansion, so that the number becomes $\ldots 000000.1000000 \ldots$ This is still cumbersome, but at least we know what it is. By convention, we drop all "leading" 0 digits before our decimal point, retaining the last one only if all non-zero digits are after the decimal place. In this case, that reduces our representation to $0.1000000 \ldots$ which is better, but still rather long. A second useful convention is to drop all trailing 0 digits after the final non-zero digit, which leaves us with

$$\frac{1}{10} = 0.1$$

as our representation.

2.2 $\frac{1}{2}$

Let us now examine the fraction $\frac{1}{2}$. How would we write this as a decimal number? We can start by recognizing that 2 is a factor of 10, and rewrite $\frac{1}{2}$ as follows:

$$\frac{1}{2} = 1 \cdot 2^{-1} = 1 \cdot 5 \cdot 5^{-1} \cdot 2^{-1} = 5 \cdot 10^{-1}$$

This then leaves us with

$$\frac{1}{2} = 0.5$$

2.3 $\frac{1}{4}$

In the case of $\frac{1}{4}$, we can rewrite the fraction as $\frac{1}{4} = \frac{25}{100}$. In this case, the decimal expansion is

$$\frac{1}{4} = \frac{25}{100} = \frac{20+5}{100} = \frac{20}{100} + \frac{5}{100} = \frac{2}{10} + \frac{5}{100} = 0.25$$

So far, we have restricted our attention to carefully chosen examples of the form $\frac{p}{q}$ where $q = 2^a 5^b$ for $a, b \in \mathbb{W}$. In these cases, if $\max(a, b)$ represents the greater (maximum) of a and b, then our fraction $\frac{p}{q}$ can always be written as

$$\frac{p \cdot 2^{\max(a,b)-a} \cdot 5^{\max(a,b)-b}}{10^{\max(a,b)}}$$

which is then easy to convert into decimal form. While this is useful for such special cases, it does not help us generate arbitrary representations of any rational number.

3 Arbitrary Decimals

Let us look at the rational number $\frac{1}{3}$ now. 3 cannot be written in the form $3 = 2^a 5^b$ for $a, b \in \mathbb{W}$. As a result, we cannot convert the denominator into the form 10^m for any $m \in \mathbb{W}$ and our above representation fails. We need to develop an algorithm that will allow us to convert any number into decimal representation, regardless of the denominator, if this is going to work.

We begin with a simple question: is $\frac{1}{3} > \frac{1}{10}$? If so, then we know[1] $a_{-1} \neq 0$. What, then, is a_{-1}? By our definition of decimal representation, we can also define a_{-1} as the smallest integer which satisfies

$$\frac{a_{-1}+1}{10} > \frac{1}{3} > \frac{a_{-1}}{10}$$

Alternatively, we can define it such that

$$\frac{a_{-1}+1}{10} - \frac{1}{3} > 0$$

and

$$\frac{1}{3} - \frac{a_{-1}}{10} > 0$$

We can find this systematically by taking our $\frac{1}{3}$ and subtracting $\frac{1}{10}$, counting the number of times we receive a positive result. Once we hit a negative result, we have exceeded a_{-1}. In this case,

$$\begin{aligned}
\frac{1}{3} - \frac{1}{10} = \frac{10}{30} - \frac{3}{30} = \frac{7}{30} &> 0 \\
\frac{7}{30} - \frac{1}{10} = \frac{7}{30} - \frac{3}{30} = \frac{4}{30} &> 0 \\
\frac{4}{30} - \frac{1}{10} = \frac{4}{30} - \frac{3}{30} = \frac{1}{30} &> 0 \\
\frac{1}{30} - \frac{1}{10} = \frac{1}{30} - \frac{3}{30} = \frac{-2}{30} &< 0
\end{aligned}$$

[1]We can see by inspection that $0 < \frac{1}{3} < 1$, so we need not examine a_i for $i \geq 0$.

so $a_{-1} = 3$.

The above process shows us that $\frac{1}{3} = \frac{3}{10} + \frac{1}{30}$. We can now find the a_{-2} such that

$$\frac{a_{-2}+1}{100} > \frac{1}{30} > \frac{a_{-2}}{100}$$

by a similar means. We can continue this process to whatever degree of accuracy we choose, finding that $a_i = 3 \forall i < 0$. Thus, this decimal representation never ends, so we say that

$$\frac{1}{3} = 0.333333333\ldots = 0.\overline{3}$$

or "zero point three repeating."[2]

3.1 Alternative Algorithm

An alternative (and possibly more convenient) algorithm also exists. Again, we begin with $\frac{1}{3}$, which we know to be less than 1. We can find a_{-1} by taking $\frac{1}{3} \cdot 10$, and then read a_{-1} as the part greater than or equal to 1. For example,

$$\frac{1}{3} \cdot 10 = \frac{1}{3} \cdot \frac{10}{1} = \frac{10}{3} = 3 + \frac{1}{3}$$

Thus, $a_{-1} = 3$. To get a_{-2}, we take the fractional part of our previous result ($\frac{1}{3}$) and multiply that by 10, taking the integer part of the answer for $a_{-2} = 3$. This method makes it easier to see that $a_i = 3 \forall i < 0$, as we always begin with the fraction $\frac{1}{3}$.

For a more complicated example, let us examine the fraction $\frac{1}{6}$. Using this algorithm, we see that

$$\frac{1}{6} \cdot 10 = \frac{1}{6} \cdot \frac{10}{1} = \frac{10}{6} = 1 + \frac{4}{6}$$

so that $a_{-1} = 1$. Continuing,

$$\frac{4}{6} \cdot 10 = \frac{4}{6} \cdot \frac{10}{1} = \frac{40}{6} = 6 + \frac{4}{6}$$

so that $a_{-2} = 6$. We are left with the same $\frac{4}{6}$ that we started with, and so we can readily see that $a_i = 6 \forall i \leq -2$, or

$$\frac{1}{6} = 0.1666666\ldots = 0.1\overline{6}$$

[2]There are at least three conventions for repeating decimal notation. The one chosen here is chosen by the author's preference, in which a line appears over all repeating digits, even if there is only one. In another convention, the line is used only for cases of at least two repeating digits, such as $0.12121212\ldots = 0.\overline{12}$, but in the case of a single digit repeating, we use a dot instead, such as $0.111111\ldots = 0.\dot{1}$. In a third convention, dots are placed above every repeating digit, as in $0.123123123\ldots = 0.\dot{1}\dot{2}\dot{3}$. All conventions are valid.

A more complicated example is $\frac{1}{7}$. For this, we have

$$\begin{aligned}
\frac{1}{7}\cdot 10 = \frac{1}{7}\cdot\frac{10}{1} = \frac{10}{7} &= 1+\frac{3}{7}\\
\frac{3}{7}\cdot 10 = \frac{3}{7}\cdot\frac{10}{1} = \frac{30}{7} &= 4+\frac{2}{7}\\
\frac{2}{7}\cdot 10 = \frac{2}{7}\cdot\frac{10}{1} = \frac{20}{7} &= 2+\frac{6}{7}\\
\frac{6}{7}\cdot 10 = \frac{6}{7}\cdot\frac{10}{1} = \frac{60}{7} &= 8+\frac{4}{7}\\
\frac{4}{7}\cdot 10 = \frac{4}{7}\cdot\frac{10}{1} = \frac{40}{7} &= 5+\frac{5}{7}\\
\frac{5}{7}\cdot 10 = \frac{5}{7}\cdot\frac{10}{1} = \frac{50}{7} &= 7+\frac{1}{7}
\end{aligned}$$

Notice that we did not see a fraction repeat until we had calculated six digits, at which point the first one finally repeated. Thus,

$$\frac{1}{7} = 0.142857142857\ldots = 0.\overline{142857}$$

This is because of our division algorithm. For $|p| < q$, each step of the algorithm will produce a result of the form

$$\frac{p}{q}\cdot 10 = a + \frac{b}{q}$$

where $0 \leq |b| \leq q$. Thus, we have a maximum of q iterations before the decimal representation either terminates ($b = 0$) or repeats (b takes on one of the $q-1$ non-zero values we have already seen.)[3]

4 Converting Decimals to Fractions

We have seen how to convert rational numbers to decimals. We now must convert decimals to rational numbers. As we shall soon see, there are only two forms of decimal numbers which we can convert into rational numbers, and some decimal numbers that cannot be converted.

4.1 Finite Decimals

Finite, or terminating decimals are the easiest to convert. These are decimals for which $a_i = 0 \forall i < N$ for some value of N. A number such as 0.125 fits this

[3]Using techniques we are not yet prepared for, we can even prove that the number of repeating digits in a repeating decimal is a factor of $q-1$. In other words, if $q = 7$, then $q-1 = 6$ and we can have 1, 2, 3 or 6 repeating digits, but not 4 or 5. We will come back to this proof when we are ready.

form, but $0.\overline{3}$ does not. Using 0.125 as the example, we have two choices. One is to begin with the full expansion and simplify, such as

$$\frac{1}{10}+\frac{2}{100}+\frac{5}{1000}=\frac{100}{1000}+\frac{20}{1000}+\frac{5}{1000}=\frac{125}{1000}$$

This is the "long" way. The easier way is to determine the denominator for the last digit in the expansion (in this case, 1000 from the $\frac{5}{1000}$ part and put all digits that follow the decimal over this number immediately, as with

$$\frac{125}{1000}$$

It is conventional to write the fraction in "lowest terms." In other words, we determine the greatest common factor of both the numerator and the denominator (possibly with the Euclidean algorithm from lesson 18) and remove this factor from both the numerator and denominator. In this case, the greatest common factor is 125, so we have

$$\frac{125}{1000}=\frac{125\cdot 1}{125\cdot 8}=\frac{1}{8}$$

as our final form.

Note the implications here: *every* finite or terminating decimal number can be written as a fraction whose denominator is of the form 10^n for some n. Thus, we can only get finite or terminating decimals from rational numbers whose denominators are of the form 2^a5^b when the fraction is in lowest terms. In all other cases, we *cannot* produce this form, and the decimal will not terminate. As we have seen above, it must then repeat. This is partly due to our choice of base 10 for our basis representation. If we used the base 8 of octal instead, the only terminating decimals would be those which had rational denominators of the form 2^a. In octal, $\frac{1}{5}=0.\overline{1463}$, for example.

4.2 Repeating Decimals

There are two types of repeating decimals to examine: those that repeat immediately after the decimal point, such as $0.\overline{1}$, and those that repeat later, such as $0.1\overline{6}$. We will deal with the first type now, and use what we learn in the second type.

In lesson 26, we learned that a geometric sequence with the initial term a, N total terms and the common ratio r has a sum given by

$$S_N=\frac{a\left(1-r^N\right)}{1-r}$$

We can use this to develop a way to convert repeating decimals into fractions. Note that, at this point, we are going to "cheat" with one of the rules of this series. We have always deliberately avoided doing derivations or proofs based on later lessons to continue developing things in order. In this case, we are going to bend that rule and use a "limit" before they are defined in lesson 31. None of the results developed in this section will be needed again until after lesson 31, so no logical inconsistencies will be created, but it will help the flow of the text in this section and keep all material related to conversion between fractions and decimals in a single lesson. Please forgive the author for bending this rule on this occasion.

The result we are going to "borrow" from our section on limits is the following: if $|r| < 1$, then

$$S_\infty = \frac{a}{1-r}$$

for an infinite series. This will be applied for the rest of this lesson, and proven rigorously in lesson 31.

4.2.1 $0.\overline{1}$

We examine $0.\overline{1}$. In this case, we have a single digit repeating immediately after the decimal place. We can express this as a geometric series with $a = r = \frac{1}{10}$. In other words,

$$0.\overline{1} = \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \left(\frac{1}{10}\right)^4 + \dots$$

Using

$$S_\infty = \frac{a}{1-r}$$

we find

$$0.\overline{1} = \frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$$

Thus, our repeating decimal has been converted into a fraction.

4.2.2 $0.\overline{63}$

The case for $0.\overline{63}$ is similar. Again, we set this up as the sum of an infinite sequence, but this time,[4] $a = \frac{63}{100}$ and $r = \frac{1}{100}$. Thus,

$$0.\overline{63} = \frac{\frac{63}{100}}{1 - \frac{1}{100}} = \frac{\frac{63}{100}}{\frac{99}{100}} = \frac{63}{99} = \frac{7}{11}$$

In general, one may notice a pattern forming. In general, if we have n repeating digits, then the fractional form of our decimal will have those n digits over a number which is formed by the digit 9 appearing n times. We then reduce the fraction to lowest terms.

4.2.3 $0.8\overline{3}$

The technique above is useful when all digits after the decimal repeat, but what if they don't? Then we must take another approach, as with $0.8\overline{3}$. In this case, the digit 8 does not repeat, but the 3 does. Let $x = 0.8\overline{3}$ to simplify notation. We have no direct process for representing x as a fraction. We can, however, represent $10x$ as a fraction quite nicely as an intermediate step that will bring us the rest of the way. Observe:

$$\begin{aligned}
x &= 0.8\overline{3} \\
10x &= 8.\overline{3} \\
10x &= 8 + 0.\overline{3} \\
10x &= 8 + \frac{3}{9} \\
10x &= \frac{75}{9} \\
x &= \frac{75}{90} \\
x &= \frac{5}{6}
\end{aligned}$$

4.2.4 The General Case

Thus, to find the fraction representation of the decimal number x which has n non-repeating digits before m repeating digits, begin with $10^n \cdot x$ and express the remaining repeating decimal as a fraction over a denominator which is an

[4] If we have n repeating digits, our denominator must be 10^n to use this technique.

m digit number with every digit 9, add the integer part to form an improper fraction, and then divide by 10^n. It sounds more complicated than it really is. One even has some flexibility in representation. For example, a masochist could treat $x = 0.\overline{1}$ as having 4 non-repeating digits 1 and 3 repeating digits 1, since $0.1111\overline{111}$ will appear exactly the same when expanded. This gives

$$\begin{aligned}
10000x &= 1111.\overline{111} \\
10000x &= 1111 + 0.\overline{111} \\
10000x &= 1111 + \frac{111}{999} \\
10000x &= 1111 + \frac{1}{9} \\
10000x &= 1111 + \frac{1}{9} \\
10000x &= \frac{10000}{9} \\
x &= \frac{1}{9}
\end{aligned}$$

4.3 The Ambiguous Case: $0.\overline{9} = 1$

There is one final example to look at: $0.\overline{9}$. This one is unusual, as it shows that decimal representation of numbers is not unique.

Using the above formulation, we can see that the bases representation expansion of $0.\overline{9}$ is an infinite series with $a = \frac{9}{10}$ and $r = \frac{1}{10}$. Using our sum formulation,

$$0.\overline{9} = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

Thus, $0.\overline{9} = 1$ and the decimal representation of that number is not unique. Similarly, $0.24\overline{9} = 0.25$, $0.4\overline{9} = 0.5$ and so forth.

5 Summary

We have a few more pieces of the puzzle. Once again, if infinite processes are permitted, then we now have a means to represent every rational number as a decimal and every terminating or repeating decimal as a fraction. What we don't have is a means to express non-terminating, non-repeating decimals as fractions. We will examine this puzzle more carefully in lesson 30.