Math From Scratch Lesson 30: Implications Of An Impossible Number

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1 Non-Terminating, Non-Repeating Decimals

In the previous lesson, we learned how to convert terminating and/or repeating decimals into fractions and vice versa. What about non-terminating, non-repeating decimals? Are these valid numbers? If so, are they rational numbers? After all, we have seen that some numbers can be represented multiple ways, such as $1 = 0.\overline{9}$. Perhaps the non-terminating and non-repeating decimals somehow "map" onto terminating and/or repeating decimals. It seems unlikely, but it's a possibility that needs to be explored.

2 Counting Numbers

We have previously seen a means to count infinite sets, and were able to demonstrate that several of the sets we have developed are *equinumerous*, meaning they are of identical size. This is counter intuitive, since many of these sets are proper subsets of others. We will review this process now.

2.1 Natural Numbers

We begin with the set of natural numbers, which is given by $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. We know that this set is infinitely large, but it is the first infinite set we've encountered, so we simply assign it a size. The conventional size is $\aleph_0$, pronounced "aleph naught," using the Hebrew letter aleph. The subscript 0 will make more sense when we get into advanced set theory.

2.2 Whole Numbers

The set of whole numbers $\mathbb{W}$ is the next set to examine. It is either the same size as the natural numbers, or it is not.[1] Instinct says it is a different size because it contains *every* natural number, *and* the number 0. The surprising result is that the addition of a finite number of entries into an infinite set does not change the size of the infinite set. Specifically, if $\aleph_0 + 1 = \aleph_0$, then the two sets are the same size.[2] The two sets are the same size if every element of $\mathbb{N}$ can be matched up with exactly one element of $\mathbb{W}$, using every number in each set. This can be done with relative ease: if $w \in \mathbb{W}$ and $n \in \mathbb{N}$, then we can map one set onto the other with $n = w + 1$. This uniquely maps every element of $\mathbb{W}$ onto an element of $\mathbb{N}$. Thus, they are the same infinitely large $\aleph_0$ in size.

2.3 Integers

We can sort our integers $\mathbb{Z}$ into a list of numbers as $0, -1, 1, -2, 2, -3, 3, -4, 4, \ldots$, sorting first by increasing absolute value and then by increasing value. With this sorting, we can see that every integer can be mapped onto a natural number as follows:

$$\begin{array}{cccccccccccc} \mathbb{Z} & : & 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & \ldots \\ & & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \ldots \\ \mathbb{N} & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \end{array}$$

Thus, the set of integers is also $\aleph_0$ in size.

[1] Yes, this is a stunning revelation, to be sure.

[2] We are a long way from dealing with such numbers in general, but the numbers which satisfy the relationship $x + 1 = x$ are the *transfinite* numbers.

2.4 Rational Numbers / Fractions

The rational numbers $\mathbb{Q}$ can also be systematically written in a list, just as the other sets can. In this case, we sort our rational numbers $\frac{p}{q} \in \mathbb{Q}$ first by the sum $|p| + |q|$ and then sort by p alone. This results in

$$\begin{array}{ccccccccccccc} \mathbb{Q} & : & \frac{0}{1} & \frac{-1}{1} & \frac{0}{2} & \frac{1}{1} & \frac{-2}{1} & \frac{-1}{2} & \frac{0}{3} & \frac{1}{2} & \frac{2}{1} & \dots \\ & & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \dots \\ \mathbb{N} & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots \end{array}$$

Thus, the set of rational numbers is also $\aleph_0$ in size.

3 Counting Non-terminating Non-repeating Decimals

To complete our examination of non-terminating, non-repeating decimals, we will check the size of the set of such numbers. Logically, the result will be one of the following cases:

1. The set will be a different size than $\aleph_0$ and, as such, it *cannot* be made exclusively from any of the sets we have previously examined.
2. The set will be $\aleph_0$ in size, and somehow map onto the rational numbers.
3. The set will be $\aleph_0$ in size, but will *not* be composed of the numbers we are familiar with.

We have already seen how to check whether or not a set is $\aleph_0$ in size by writing its elements in a sorted list that can be matched against the set of natural numbers, so that will be our first test. If the size is, indeed, $\aleph_0$, then we'll need to develop a new test to distinguish between the last two cases.

We will determine that we are actually in case 1 using proof by contradiction: we will show that it is, in fact, impossible to create a complete list of non-terminating, non-repeating decimal numbers. If the list cannot be created, the set cannot be $\aleph_0$ in size, and thus it cannot be another means of representing any of the sets we have seen previously. We will either be dealing with a new set of numbers entirely, or we will be dealing with meaningless gibberish.

We employ proof by contradiction by assuming we already have a complete list of non-terminating, non-repeating decimals greater than 0 and less than 1 which have been sorted in some fashion. We label the nth entry in the list as r_n, thus ensuring that we have a match to the natural numbers n.

$$
\begin{aligned}
r_1 &= 0.123456789101112\ldots \\
r_2 &= 0.141421356237309\ldots \\
r_3 &= 0.314159265358979\ldots \\
r_4 &= 0.173205080756887\ldots \\
r_5 &= 0.223606797749968\ldots \\
r_6 &= 0.244948974278317\ldots \\
r_7 &= 0.264575131106459\ldots \\
r_8 &= 0.282842712474619\ldots \\
r_9 &= 0.316227766016837\ldots \\
&\vdots
\end{aligned}
$$

If we can somehow prove that this list is not complete, then we can prove that it is not complete. If the "complete" list is not complete, then no such list can exist, the non-terminating, non-repeating decimals cannot be mapped to the natural numbers and vice versa, and this set of numbers is something completely different.

We construct the number r_0 as follows: r_0 is a non-terminating, non-repeating decimal which is less than 1 and greater than 0. As with all decimals in this range, it can be written in the form

$$r_0 = \sum_{i=1}^{\infty} a_i \cdot 10^{-i}$$

by the bases representation theorem. We define each a_i by forcing it to be different from the corresponding digit in r_i. Rewriting our list with the relevant digits in bold blue font, we are looking at the following:

$$\begin{aligned}
r_1 &= 0.\mathbf{1}23456789101112\ldots \\
r_2 &= 0.1\mathbf{4}1421356237309\ldots \\
r_3 &= 0.31\mathbf{4}159265358979\ldots \\
r_4 &= 0.173\mathbf{2}05080756887\ldots \\
r_5 &= 0.2236\mathbf{0}6797749968\ldots \\
r_6 &= 0.24494\mathbf{8}974278317\ldots \\
r_7 &= 0.264575\mathbf{1}31106459\ldots \\
r_8 &= 0.2828427\mathbf{1}2474619\ldots \\
r_9 &= 0.31622776\mathbf{6}016837\ldots \\
&\vdots
\end{aligned}$$

Algebraically, if we say that

$$r_i = \sum_{j=1}^{\infty} b_j \cdot 10^{-j}$$

then we can define each a_i as

$$a_i = \begin{cases} b_j + 1 & b_j < 9 \\ 0 & b_j = 9 \end{cases}$$

With our above example, this gives us

$$r_0 = 0.255319227\ldots$$

This number *cannot* appear on our list: it differs from each r_i by at least the ith digit after the decimal. Thus, our list is *not* complete. The only assumption we made about this list is that it was complete, so this logical contradiction must mean our assumption is incorrect. There cannot be a complete list of non-terminating, non-repeating decimals. If such a list does not exist, then we have two possibilities:

1. Everything we've done from the introduction of infinite processes until now has been a waste of time and must be discarded.
2. We need to extend our set of numbers in some fashion so that all of that work has meaning, but risk losing some of our algebraic axioms in the context of infinite processes, such as closure, unless we can find a suitably broad definition of "number" to preserve as many of these features as possible.

We will take the second route. Lesson 31 will introduce the notion of a *limit*, and lesson 32 will use that notion in conjunction with most of volume 2 to define the set of real numbers.