Math From Scratch Lesson 31: Limits

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1 More Possibilities

For the past few lessons, we have been examining a single choice. Do we allow infinite processes? The answer ultimately boils down to personal choice for the task at hand. If infinite processes are allowed, some axioms are lost. If they are not allowed, then some operations are not allowed within our algebra in all cases. This lesson establishes the final major advantage of allowing infinite processes. With this advantage in place, we can finally define the real number system in a formal way.

2 Limits

When examining an infinite process, one would prefer that it "converges," or comes up with a single, useful answer. Since we cannot actually perform an infinite process an infinite number of times, we need to find a way to test the process and see if a single number will result. This test is known as "taking the limit" of the process. Taking a limit will involve two arbitrary numbers: epsilon (ϵ) and N.

2.1 Step 1: Expressing the Infinite Process as a Sequence

As our first example, let us take the sequence defined by

$$x_n = 1 - \frac{1}{n}, n \geq 1$$

Computing the first few terms by hand, we find that

$$\begin{aligned}
x_1 &= 1 - 1 = 0 \\
x_2 &= 1 - \frac{1}{2} = \frac{1}{2} \\
x_3 &= 1 - \frac{1}{3} = \frac{2}{3} \\
x_4 &= 1 - \frac{1}{4} = \frac{3}{4} \\
x_5 &= 1 - \frac{1}{5} = \frac{4}{5} \\
x_6 &= 1 - \frac{1}{6} = \frac{5}{6} \\
x_7 &= 1 - \frac{1}{7} = \frac{6}{7} \\
x_8 &= 1 - \frac{1}{8} = \frac{7}{8} \\
x_9 &= 1 - \frac{1}{9} = \frac{8}{9} \\
x_10 &= 1 - \frac{1}{10} = \frac{9}{10} \\
&\vdots \\
x_n &= \frac{n-1}{n}
\end{aligned}$$

There is no specific value of n for which $x_n = 1$. If there were, then we would have $\frac{n-1}{n} = 1$ or $n - 1 = n$, which leads to $-1 = 0$ when n is canceled. This is impossible. As we shall soon see, however, this infinite process has a *limit* of 1. The sequence can get as close to 1 as we want, but it will never exactly equal 1. The question is, how do we define "close" in this context? That's where step 2 comes in.

2.2 Step 2: Selecting Epsilon ϵ

The arbitrary number epsilon (ϵ) is the one that we use to define "close." We say that the sequence has a limit of L if and only if $|x_n - L| < \epsilon$ for some $n > N$. In other words, we decide how close "close" is and set that value as ϵ, and make sure that x_n is at least that close any time that we go "far enough" into the sequence. Now, "far enough" is a number we calculate, and not one we choose. We also need to recognize that L is a limit if and only if this holds regardless of how small ϵ is. With our example sequence, one could try to claim that the number 2 is a valid limit with $\epsilon = 3$, simply because *every* term in the sequence is within 3 units of 2. This calculation is as follows, recognizing that $2 > x_n$ for every n, so

$$\begin{aligned} 2 - x_n &< 3 \\ 2 - \left(1 - \frac{1}{n}\right) &< 3 \\ 2 - 1 + \frac{1}{n} &< 3 \\ 1 + \frac{1}{n} &< 3 \\ \frac{1}{n} &< 2 \end{aligned}$$

which is always true for $n \geq 1$. If, however, we choose $\epsilon = 0.1$, then we find that

$$\begin{aligned} 2 - x_n &< 3 \\ 2 - \left(1 - \frac{1}{n}\right) &< 0.1 \\ 2 - 1 + \frac{1}{n} &< 0.1 \\ 1 + \frac{1}{n} &< 0.1 \\ \frac{1}{n} &\nleq -0.9 \end{aligned}$$

which is not true: therefore, 2 is not a valid limit.

If the limit is valid, we will end with an equality of the form $n > N(\epsilon)$ in one form or another. $N(\epsilon)$ is allowed to be a function of ϵ, provided N gets larger as ϵ gets smaller.[1]

[1] When we get to calculus we will see a formal test of this property, but to get there, we must first establish limits. For now, we will have to settle for using a definition that can be verified by inspection alone, using the simple fact that $\frac{1}{x}$ gets larger as x gets smaller.

2.3 Step 3: Applying the Test and Solving for N

Here we test to see if 1 truly is the limit for this sequence. We set up our inequality, dropping the absolute value in favor of recognizing that $1 > x_n \forall n$.

$$\begin{aligned} 1 - x_n &< \epsilon \\ 1 - \left(1 - \frac{1}{n}\right) &< \epsilon \\ 1 - 1 + \frac{1}{n} &< \epsilon \\ \frac{1}{n} &< \epsilon \\ n &> \frac{1}{\epsilon} \end{aligned}$$

Thus, if we want $\epsilon = 0.1$, we need only choose $N > 10$ and the result is ensured. If we want to be closer, such as $\epsilon = 0.01$, then we choose $N > 100$. For whatever epsilon we choose, we calculate $\frac{1}{\epsilon}$ and then arbitrarily choose any number N which is greater than that fraction. With this definition of N, which works for arbitrarily small epsilon ϵ, we can establish the limit L.

3 Complete Sets

In lesson 28, we showed that the infinite sequence

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ x_1 &= 1 \end{aligned}$$

approaches $\sqrt{2}$, a number which does not belong in any set of numbers we have seen thus far. Of course, we couldn't confirm that this was even a valid process at the time. We still can't, because we have the flexibility to decide if the process is valid or not. That choice comes in the form of a thirteenth axiom.

A set $\mathbb{S}$ is complete if and only if every convergent series using elements of $\mathbb{S}$ converges to an element of $\mathbb{S}$. Thus, $\mathbb{Q}$ is not a complete set; the above sequence satisfies $x_n \in \mathbb{S}$ for every finite n, but the limit $(\sqrt{2})$ is not an element of $\mathbb{S}$. This will be significant in the next lesson.

4 Followup: Proving the Infinite Geometric Series Formula

As promised in lesson 29, we will now prove the validity of the formula for the sum of an infinite geometric series. First, recall that a geometric sequence is one in which

$$t_n = ar^{n-1}$$

for some a and r. A geometric series is one in which

$$S_n = \sum_{i=1}^{n} t_n$$

We used the formula

$$S_\infty = \frac{a}{1-r}$$

without proof. We shall prove the formula now.

The proof begins with the formula that we derived for the sum of a finite geometric series:

$$S_N = \frac{a\left(1-r^N\right)}{1-r}$$

We are now going to show that, provided $|r| < 1$, the infinite series has a finite limit by finding a value for N that works for an arbitrarily small ϵ. We initially set up our inequality as follows:

$$\begin{aligned}
\epsilon &> |x_n - L| \\
&> \left|\frac{a\left(1-r^N\right)}{1-r} - \frac{a}{1-r}\right| \\
&> \left|\frac{a\left(1-r^N\right) - a}{1-r}\right| \\
&> \left|\frac{a - ar^N - a}{1-r}\right| \\
&> \left|\frac{-ar^N}{1-r}\right| \\
&> \frac{\left|-ar^N\right|}{|1-r|} \\
&> \frac{\left|ar^N\right|}{|1-r|} \\
\epsilon\,|1-r| &> \left|ar^N\right|
\end{aligned}$$

Now, if we have assumed that $|r| < 1$, we can show that the left hand side is always positive. This leaves us with two cases to check. If $ar^N < 0$, the statement is proven and we are done. If $ar^N > 0$, we have a little more work to do.

When both $a > 0$ and $r > 0$, we get

$$r^N < \frac{\epsilon\,(1-r)}{a}$$

In this case, the right hand side is an arbitrarily small constant. As long as $|r| < 1$, r^N will get smaller and smaller as N increases. Thus, there is such an N. We can't yet solve specifically for N (as we are a *long* way from logarithms) but we can see that such an N exists. Finding a functioning N from this point is a matter of patience and intelligent guesswork.

When both $a < 0$ and $r < 0$, we find that any even value of N will do, and that we arrive at the same inequality

$$r^N < \frac{\epsilon\,(1-r)}{a}$$

for odd N.

5 Next Lesson

In our next lesson, we finally define the real numbers using the 13 required axioms.