Math From Scratch Lesson 32:
The Real Numbers

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1 The Real Numbers

The past few lessons have been spent exploring the usefulness and validity of infinite processes. In the end, including them is up to us. Without infinite processes, we have already established the entirety of the available math, and can explore anything with rational numbers and a finite number of steps. To develop any math beyond this, we must choose to include infinite processes and move into a new system. This new system has axioms which fail for infinite processes, but has a new axiom which allows infinite processes to exist in other logical situations. The 13 axioms which define the real numbers $\mathbb{R}$ are as follows:

1. **Closed under +** $\forall x, y \in \mathbb{R} x + y \in \mathbb{R}$
2. **Closed under** $\cdot$ $\forall x, y \in \mathbb{R} x \cdot y \in \mathbb{R}$
3. **Commutes under +** $\forall x, y \in \mathbb{R}, x + y = y + x$
4. **Commutes under** $\cdot$ $\forall x, y \in \mathbb{R}, x \cdot y = y \cdot x$
5. **Associative under +** $\forall x, y, z \in \mathbb{R}, x + (y + z) = (x + y) + z$
6. **Associative under** $\cdot$ $\forall x, y, z \in \mathbb{R}, x \cdot (y \cdot z) = (x \cdot y) \cdot z$
7. **Identity under +** $\exists 0 \in \mathbb{R} : \forall x \in \mathbb{R} x + 0 = x$

8. **Identity under** $\cdot$ $\exists 1 \in \mathbb{R} : \forall x \in \mathbb{R} x \cdot 1 = x$
9. **Inverses under +** $\forall x \in \mathbb{R} \exists -x : x + (-x) = 0$
10. **Inverses under** $\cdot$ $\forall x \neq 0 \in \mathbb{R} \exists x^{-1} : x \cdot x^{-1} = 1$
11. **Distributive Property** $\forall x, y, z \in \mathbb{R} x \cdot (y + z) = x \cdot y + x \cdot z$
12. **Axiom of Inequality** $\forall x, y, z \in \mathbb{R}$ if $x < y$ and $y < z$ then $x < z$
13. **Completeness Axiom** This is the new axiom. This axiom states that every convergent sequence x_n where $x_n \in \mathbb{R} \forall n$ converges to a limit $L \in \mathbb{R}$. In other words, if the limit of a sequence of real numbers exists, then that limit is also a real number.

It is the final axiom which defines the real numbers. This is why the set of real number is sometimes referred to as the *closure* of the rational numbers: we can create the real numbers by taking the rational numbers and then "closing" the holes that exist in and around the limits to its infinite sequences. This process also defines a new set of numbers which we will refer to in the future.

2 Irrational Numbers

The irrational numbers $\overline{\mathbb{Q}}$ are defined as the set of real numbers that are not rational numbers, or $\overline{\mathbb{Q}} = \mathbb{R} \setminus \mathbb{Q}$. This is where the symbol comes from: the line above a symbol means "not" in logic, so the symbol $\overline{\mathbb{Q}}$ quite literally means "not rational." The irrational numbers we have encountered thus far are the square roots of imperfect squares. For example, $\sqrt{2} \in \overline{\mathbb{Q}}$, since it is not a rational number but it is the limit of a sequence of rational numbers.

In lesson 30, we showed that the set of all possible decimal numbers is too large to be limited to the set of rational numbers. The irrational numbers include all possible decimal numbers that aren't rational numbers. That is a challenge to prove directly, but it can be proven indirectly as follows: every possible decimal number can be represented as an infinite sequence of rational numbers, where each term in the sequence adds one digit to the eventual representation. This sequence will converge to a limit: for whatever ϵ we choose, we can choose an N such that the differences between iterations are more decimal places down the line than we'd "see" if we go to the final digit for ϵ. (If ϵ is chosen to be $\frac{1}{7}$ or some other repeating, non-terminating decimal, we simply use the bases representation theorem to convert our number from base 10 to a base which is a multiple of the denominator in ϵ, converting ϵ into a terminating decimal and finishing our proof in that base. Because we will be using ϵ in our algebra, it must be a rational number, and can therefore always be expressed as a fraction.)

Thus, the decimal we are looking at is a real number, since the limit would be defined. If that number is not a rational number, then by definition, it must be irrational.

The set of irrational numbers does not form any algebra we have defined. It does not contain either identity $(0, 1 \in \mathbb{Q})$ and is not closed under either addition or multiplication $(\sqrt{2} + \left(-\sqrt{2}\right) = 0,\ \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1)$. It does satisfy the other 9 axioms of the real numbers listed above, but without closure, it cannot be used to form an algebra.

The set of irrational numbers is also significantly larger than the set of rational numbers. As we saw in lesson 30, the set of all decimals is "larger" than the set of rational numbers. We will (eventually) be able to show that, if a set of order x is not complete, then the order of its closure is 2^x. Furthermore, we will be able to show that there are no sets of order y such that $x < y < 2^x$. This means that the set of irrational numbers is of order that is either $\aleph_0$, just as the rational numbers, or it is of order $2^{\aleph_0}$, just as the set the real numbers is. Since it must be larger than $\mathbb{Q}$, that forces us to conclude that it is the same size as $\mathbb{R}$.

3 The Future of the Series

With the real numbers established, we can technically dive right into calculus. I find, however, that it would be beneficial to graph functions before getting into calculus, since that greatly improves the understanding of the topic. There are two options.

1. Dive in with Euclid's axioms of geometry, build geometry from there, and then move to graphing.
2. Try to get to graphing with a minimum of new axioms. To do this, instead of adding the five axioms needed for Euclidean geometry, we add two axioms for vector spaces, and remove some of the axioms we are currently using. We will develop polynomials and polynomial algebra, solving for roots, bringing the complex numbers into play to find all possible roots. Equipped with these tools, we can define matrices and determinants, needed to complete the work on vector spaces, which, in turn, allows us the freedom to define vectors. Vectors provide inner products and inner product spaces, and can eventually get us to directions. Once we have directions, then we can start to plot a pair of axes, justify why they must be perpendicular, and move into graphing from there.

The first option is clearly the more expedient. The goal of *Math From Scratch*, however, is to build as much mathematical framework as possible using as few axioms as possible. Thus, we will be taking the latter route. Volume three will, be necessity, be quite large.