Math From Scratch Lesson 34: Isolating Variables

W. Blaine Dowler

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1 Order of Operations

1.1 Multiplication and Addition

The rules for algebra that we have laid out are very flexible, and allow for a variety of operations and very complicated expressions and equations. When we use these equations to represent the physical world, as is the foundation of much of science, then we may need to determine the value of a single variable at any time. To do so, we'll need a means to perform calculations clearly. After all, expressions like $1 + 2 \times 3$ can be ambiguous: if we perform addition first, then $1 + 2 \times 3 = 3 \times 3 = 9$, but performing multiplication first results in $1 + 2 \times 3 = 1 + 6 = 7$. Since a single expression is useless if it cannot be consistently calculated[1], we need to determine the best way to perform these operations.

[1]In some cases, this is unavoidable, but there will be strict regulations dealing with these ambiguities

We know that addition and multiplication are both commutative. We wish to look for the most consistent way to calculate $1 + 2 \times 3$, $2 \times 3 + 1$, $1 + 3 \times 2$, $3 \times 2 + 1$, $2 + 1 \times 3$ and $1 \times 3 + 2$. Let us see what the results are for each arrangement of numbers with each operation.[2]

Expression	**Addition First**	**Multiplication First**
$1 + 2 \times 3$	9	7
$2 \times 3 + 1$	8	7
$1 + 3 \times 2$	8	7
$3 \times 2 + 1$	9	7
$2 + 3 \times 1$	5	5
$1 \times 3 + 2$	5	5

Doing addition first results in three different results, appearing twice each. Doing multiplication first is much more consistent, with the result 7 appearing four times and the result 5 appearing twice. Furthermore, all arrangements that result in the answer 5 are only valid rearrangements if we calculate addition before multiplication. Thus, if multiplication is calculated before addition, the last two rows of the table are invalid arrangements, and all results are a consistent value of 7. Therefore, if multiplication must be performed before addition, then we have a wholly consistent system.

1.2 Division and Subtraction

We must also find places to put the operations of division and subtraction in this sequence. Since they are formally defined in relation to multiplication and addition, this is not difficult: division gets the same priority as multiplication, and subtraction gets the same priority as addition.

1.3 Exponents

The final major operation to date is exponentiation. Again, we can have a seemingly ambiguous case with expressions such as 2×3^4. If we perform multiplication first, we get $2 \times 3^4 = 6^4 = 1296$, but if we do the exponentiation first, we get $2 \times 3^4 = 2 \times 81 = 162$, which is a very different result. In this case, we resolve the issue through the definition of the exponent: writing this out in totality, we find that $2 \times 3^4 = 2 \times 3 \times 3 \times 3 \times 3 = 162$. Therefore, it is clear that

[2] One may object to some of these arrangements, allowing only four of these arrangements instead of six. That is a side effect of an instinctive grasp of the order of operations that we have been trained to apply; there is no intrinsic meaning to these systems that makes the answer clear.

exponents must be calculated before multiplication, since that is the consistent result.

1.4 Brackets

There are times in which we would prefer to prioritize the later operations. For example, if we want to determine the total revenue collected from sales in Dollar Store, it is preferable to add up the items sold first, and then multiply by the revenue per item second, avoiding a much lengthier calculation if each item's price is multiplied in advance. We can use brackets as the signal for this, as we did with the distributive property:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Any calculations written in brackets are to be performed before calculations written outside of the brackets. Here, on the left, we compute $b + c$ before multiplying anything.

1.5 The Resulting Acronym

The resulting acronym commonly taught as a mnemonic for this process is expressed in two ways, depending upon the region of instruction:

- **PEMDAS** - This stands for "parentheses, exponents, multiplication, division, addition, subtraction."
- **BEDMAS** - This stands for "brackets, exponents, division, multiplication, addition, subtraction."

In both cases, the final result is the same, as multiplication and division are interchangeable (by virtually of really being the same operation) and "parentheses" is a synonym for "brackets." This is the order we use to calculate these expressions. For example, in

$$d = \frac{(v_f - v_i)}{t}$$

we compute the value of d by first calculating the bracketed $v_f - v_i$, and then dividing the result by t. It is customary to imply brackets using fraction notation:

$$d = \frac{v_f - v_i}{t}$$

We know that the $v_f - v_i$ gets computed before the division because the line in a fraction (named the *vinculum*) extends beneath the entire fraction. In general, we should always read the fraction $\frac{a}{b}$ as $(a) \div (b) = (a) \cdot (b)^{-1}$ to avoid ambiguity.

2 Isolating Variables

Calculating d in

$$d = \frac{v_f - v_i}{t}$$

is all well and good, but what if the value of d is known and we need to find the value of v_i? That requires rearranging variables to isolate v_i. The trick to this is to perform our order of operations (BEDMAS/PEMDAS) in reverse order.

Expressing this as

$$d = (v_f - v_i) \div t$$

gives us the first step in the process. Let us colour code what we are looking for:

$$d = (v_f - {\color{blue}v_i}) \div t$$

It is currently contained within brackets, so our order of operations for isolating variables (SAMDEB/SADMEP) needs to deal with the $\div t$ portion first. We deal with it by finding a way to cancel it out, using what we know of cancelable elements from our earlier lessons. If we multiply by t, this will eliminate the $\div t$ portion. If we multiply by t on the right, we must also do so on the left; failing to do so would mean the two sides are not equal any longer. Thus, we have

$$\begin{aligned} d \cdot t &= (v_f - {\color{blue}v_i}) \div t \cdot t \\ d \cdot t &= (v_f - {\color{blue}v_i}) \\ d \cdot t &= v_f - {\color{blue}v_i} \end{aligned}$$

Now that we are down to the interior of the brackets, we run through (SAMDEB/SADMEP) again. The v_f term is being added, so we eliminate it by subtracting v_f again, which is the same as adding $-v_f$ to both sides. Thus, we have

$$\begin{aligned} d \cdot t &= v_f - {\color{blue}v_i} \\ -v_f + d \cdot t &= -v_f + v_f - {\color{blue}v_i} \\ d \cdot t - v_f &= -{\color{blue}v_i} \end{aligned}$$

Our final step is to multiply by -1 to completely isolate v_i.

$$\begin{aligned} -1\,(d \cdot t - v_f) &= -1 \cdot (-{\color{blue}v_i}) \\ v_f - d \cdot t &= {\color{blue}v_i} \end{aligned}$$

Thus,

$$v_i = v_f - d \cdot t$$

and the variable is isolated. We will run into the aforementioned ambiguities when we deal with solving for variables that appear with exponents. As it turns out, there are not ambiguous cases, but rather are cases that legitimately have more than one correct answer.

3 Next Lesson

In our next lesson, we begin to examine polynomials and their roots, which will eventually lead us to the realization that the real numbers alone are not adequate to describing the human experience.