Math From Scratch Lesson 36: Polynomial Roots

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1 Polynomial Roots Defined

Although polynomials themselves form an algebraic ring rather than a field, they do map onto the real numbers when you have specific values for the parameters a_n and the variable x. A specific value x_1 is considered a "root" of $P(x)$ if and only if $P(x_1) = 0$, given the particular values of the various a_n. Roots are only effectively discussed if the order of the polynomial is 1 or higher; the specific value of x has no bearing on the constant term which makes up zero order polynomials.

2 Roots of First Order Polynomials

If we have a polynomial of order 1, also referred to as linear polynomials, then we have $P(x) = a_0 + a_1 x$ with $a_1 \neq 0$. The parameter a_0 can take on any real value. To determine if this polynomial has a root, and what that root may be, we need to find the value of x_1 in terms of a_0 and a_1 such that $P(x_1) = 0$.

We start by assuming the root exists, and work to isolate x_1 just as we have in the past:

$$\begin{aligned} 0 &= a_0 + a_1 x_1 \\ -a_0 &= a_1 x_1 \\ -\frac{a_0}{a_1} &= x_1 \end{aligned}$$

If we substitute this value back into $P(x)$, we get

$$P(x_1) = a_0 + a_1 \left(-\frac{a_0}{a_1} \right) = a_0 - a_0 = 0$$

Thus, we have found the root of the polynomial in its general form. Since we know that $a_1 \neq 0$, this is defined for all possible first order polynomials.

3 Roots of Second Order Polynomials

It is in our pursuit of the roots of general second order polynomials that we will, eventually, find ourselves discarding one of the axioms of the real numbers to extend the definition of valid numbers into a much broader definition, and that, in turn, will eventually lead us to graphing.

Let us examine a general second order polynomial, also referred to as a quadratic polynomial. Instead of subscripting various a_n terms, we will instead use the more common

$$P(x) = ax^2 + bx + c$$

There are a number of approaches that we can take to determine the general solution in this case. We start by looking for specific and simple cases, and work ahead from there.

3.1 Case 1: $b = 0$

Let us begin with the case where $b = 0$. This reduces the polynomial to

$$P(x) = ax^2 + c$$

We can find any roots with relatively simple algebra at this point:

$$\begin{aligned} 0 &= ax^2 + c \\ -c &= ax^2 \\ -\frac{c}{a} &= x^2 \\ x &= \pm\sqrt{-\frac{c}{a}} \end{aligned}$$

where $\pm$ means we actually have *two* roots: one at $x = \sqrt{-\frac{c}{a}}$ and another at $x = -\sqrt{-\frac{c}{a}}$.

This is where the issues mention above start to come into play: if $-\frac{c}{a} < 0$, then there are no known values which satisfy the expression $\sqrt{-\frac{c}{a}}$. We will eventually see that the axiom of inequality is not necessary, and that removing said axiom will allow us to solve questions of this type.

3.2 Case 2: Perfect Square Quadratics

There is another special case that is fairly easy to work with. If our polynomial is of the form

$$P(x) = a^2x^2 + 2abx + b^2$$

then we can find the roots with relative easy. If we apply the definition of polynomials that we have above to

$$(ax + b) \cdot (ax + b)$$

we can see that

$$(ax + b) \cdot (ax + b) = a^2x^2 + abx + bax + b^2 = a^2x^2 + 2abx + b^2$$

which is the form above. Therefore, our second degree polynomial has two factors, each of which is $ax+b$. If we calculate $ax+b$ for each a, b and x then we get a real number for an answer. The only way that $P(x) = (ax + b)\cdot(ax + b) = 0$ can be true is if $ax + b = 0$ on its own, which is a case we have already solved. Thus, the root of

$$P(x) = a^2x^2 + 2abx + b^2$$

is $x = -\frac{b}{a}$. We know that $a \neq 0$, otherwise this wouldn't be a second degree polynomial.

3.3 Case 3: Difference of Perfect Squares

The third special case is that of the difference of perfect squares. If the polynomial is of the form

$$P(x) = a^2x^2 - b^2$$

then we can also easily find the roots. Let us examine the product $(ax + b) \cdot (ax - b)$ in detail.

$$(ax + b) \cdot (ax - b) = a^2x^2 + abx - abx - b^2 = a^2x^2 - b^2$$

Thus, this polynomial can be factored as well. If the entire polynomial is 0, then either one factor is zero, or the other factor is zero. Therefore, there are two roots for this as well: $x_\pm = \pm\frac{b}{a}$. Both are valid roots, as we can see by explicit substitution:

$$P(x_+) = a^2\left(\frac{b}{a}\right)^2 - b^2 = b^2 - b^2 = 0$$

and

$$P(x_-) = a^2\left(-\frac{b}{a}\right)^2 - b^2 = b^2 - b^2 = 0$$

This differs from the first case only in the sense that we have perfect squares, so we need not worry about the square roots of negative numbers.

3.4 Case 4: Other Factorable Forms

The next special case is one that is easier to develop by working from the result than from the original form. Thus far, we have been able to factor polynomials with some sort of symmetry in the factors. If we have two unrelated factors, such as $ax + b$ and $cx + d$, then the quadratic polynomial they produce is of the form

$$P(x) = (ax + b)(cx + d) = acx^2 + adx + bcx + bd = acx^2 + (ad + bc)x + bd$$

Demonstrating that $-\frac{b}{a}$ and $-\frac{d}{c}$ are roots is straightforward. (Again, we know that $a \neq 0$ and $c \neq 0$ as either of those conditions would mean this is not a quadratic.) The challenge here is to identify the quadratics which are of this form.

Let us use concrete examples to develop the algorithms involved. Let us begin with

$$P(x) = x^2 + 7x + 10$$

If we compare this to our general form, we see that $ac = 1$. Thus, either $a = c = 1$ or $a = c = -1$. We will start with the assumption $a = c = 1$ for simplicity; we will be able to show that the roots we develop in either case are identical.

With $a = c = 1$, we compare the other coefficients. The linear term (i.e. the term with x) $b + d = 7$, and the constant term is $bd = 10$. We can attempt to solve this algebraically, by using the first equation to isolate one variable and substituting this expression into the second, but this will only result in a new quadratic polynomial, which is hardly of any use. At this point, we need to work by inspection. The fact that $bd = 10 > 0$ means that b and d are either both positive, or both negative. The factors of 10 are $\pm 1, \pm 2, \pm 5$ and ± 10. The linear term 7 is also positive. Since this is the sum of b and d, they must both be positive numbers. We look to the list of factors of 10 to see if any positive pair add up to 7. In fact, 2 and 5 fit the bill. We can verify that $(x + 2)(x + 5) = x^2 + 7x + 10$, making the roots -2 and -5.

If we had chosen $a = c = -1$, we would still see that b and d were either both positive or both negative thanks to the fact that $10 > 0$, but our linear term would have been $-b - d = 7$, thereby ensuring that b and d were both negative. This would give us the factorization $(-x - 2)(-x - 5) = x^2 + 7x + 10$ and the same set of roots.

The complete algorithm for factoring a polynomial of the form

$$P(x) = ax^2 + bx + c$$

with hopes for factors with integer coefficients is as follows:

1. Identify factors of a
2. Identify factors of c
3. Check the sign of c. If it is positive, then the signs of the constant terms in the linear factors will match each other. They may be either both positive or both negative, but they'll match. If the sign of c is negative, then the signs in the linear factors will be different.
4. Check the sign of b. If the sign of c was positive, then the sign of b is is the same as the signs of the constants in the linear terms. If the sign of c was negative, then the sign of b will match the sign of the constant term in the linear factors which leads to the largest absolute magnitude in its product. That is, in the notation $P(x) = acx^2 + (ad + bc)x + bd$, this sign will match the sign of either ad or bc, whichever product has the largest absolute magnitude.
5. Look at the lists of factors of a and c and identify which, if any, produce this particular combination.

There are polynomials which cannot be solved in this manner. For example, take

$$P(x) = x^2 + x + 1$$

as our polynomial. All coefficients are 1, so there are only four possibilities that can produce the quadratic (x^2) term and constant (1) term:

$$\begin{aligned}
(x+1)(x+1) &= x^2 + 2x + 1 \neq P(x) \\
(x-1)(x-1) &= x^2 - 2x + 1 \neq P(x) \\
(-x-1)(-x-1) &= x^2 + 2x + 1 \neq P(x) \\
(-x+1)(-x+1) &= x^2 - 2x + 1 \neq P(x)
\end{aligned}$$

None of these work. Therefore, there are no solutions with integer coefficients in these factors. We need to look for a general form of the solution if we are going to be successful here.

3.5 The General Case: Completing the Square

Let us look only at the form

$$P(x) = ax^2 + bx + c$$

If we had something like one of our first four cases, it would be easy. We'll use a technique officially named "completing the square," but which the author informally calls an application of "the method of wishful thinking." We wish that this is a perfect square for case 2, so we find a way to make it work.

We start by factoring out the a from the first two terms. There is no explicit a in the second term, so we get a balancing term in the denominator:

$$P(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

The term in brackets is similar to a perfect square. The factor $\left(x + \frac{b}{2a}\right)$, when multiplied by itself, produces the quadratic

$$\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

This is $\frac{b^2}{4a^2}$ higher than the term in the brackets above. This is where the wishful thinking comes in: we force this to match the above form by adding a

$\frac{b^2}{4a^2}$ term. Doing so changes the polynomial, unless we balance it by adding a matching term of $-\frac{b^2}{4a^2}$ as well. Thus,

$$P(x) = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + 0\right) + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

Now we take the additional term of $-\frac{b^2}{4a^2}$ out of the brackets through multiplication by a:

$$P(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$

We how have a piece in the middle that is a perfect match to the difference of perfect squares. Namely,

$$P(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

We can now look for roots by isolating x, since x appears only once in this equation.

$$\begin{aligned}
a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c &= 0 \\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} + \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

We now have the general solution, often referred to as the *quadratic equation formula*. Because we left a, b and c as parameters, rather than substituting any specific value, we can use and reuse the final line of this result as often as we like. We need not be concerned about the restriction that $a \neq 0$, for that is assumed by the fact that we are using second order polynomials. The greater concern is the presence of $\sqrt{b^2 - 4ac}$. This is not defined when $b^2 - 4ac < 0$. In fact, this is called the *discriminant* since it can discriminate between three different cases:

- If $b^2 - 4ac > 0$, we will have two identifiable and distinct roots.
- If $b^2 - 4ac = 0$, we will have a single root identifiable through this process.
- If $b^2 - 4ac < 0$, we will be unable to identify any roots amongst the real numbers.

4 Final Remarks

We have yet to make any comment about counting all possible roots for a given polynomial. In fact, we cannot do so at this stage: it will not be until after we discard the axiom of inequality that we will be able to prove definitively that we can count the roots to such polynomials as these, and it is at that stage that we will do so, via the Fundamental Theorem of Algebra. At the present time, we cannot prove that any given polynomial does not have roots in addition to those we have identified above. We shall continue to identify general methods to find roots for polynomials of order 3 and 4, before discarding the axiom of inequality to work in truly general cases. The reasons we stop at order 4 and do not continue to orders 5 and higher will have to wait much longer than that.

5 Next Lesson

Next time, we examine the methods to solve cubic polynomial equations, which are polynomials of order 3, in much greater detail.