Math From Scratch Lesson 37: Roots of Cubic Equations

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September 1, 2013

Contents

1 Defining Cubic Equations

A cubic equation is a third order polynomial equation. In our standard notation, it is denoted

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

2 The Roots of Cubic Equations

We will now find the roots of this equation, starting again with a special case.

2.1 Case 1: $a_2 = a_1 = 0$

In this case, our polynomial becomes

$$P(x) = a_3 x^3 + a_0$$

The roots of this are found by

$$\begin{aligned} 0 &= a_3 x^3 + a_0 \\ -a_3 x^3 &= a_0 \\ x^3 &= -\frac{a_0}{a_3} \\ x &= -\sqrt[3]{\frac{a_0}{a_3}} \end{aligned}$$

2.2 Case 2: $a_2 = 0$

In the case for which $a_2 = 0$, our polynomial reduces to

$$P(x) = a_3 x^3 + a_1 x + a_0$$

When solving for the roots of this equation, we set it equal to zero:

$$0 = a_3 x^3 + a_1 x + a_0$$

To simplify the process, we will start by dividing by a_3. If we set $a = \frac{a_1}{a_3}$ and $b = \frac{a_0}{a_3}$ (which we can do, as $a_3 \neq 0$, since this would otherwise only be a quadratic equation) then this becomes

$$0 = x^3 + ax + b$$

This changes our coefficients from integers into rational numbers, but the algebra itself is simplified. We begin with a change of variables of the form $x = u - v$. We will be able to choose our own u and v so long as they conform to this constraint, so the process and result will be valid as long as this is still true in the finished product.

Our equation now becomes

$$
\begin{aligned}
0 &= (u-v)^3 + a(u-v) + b \\
&= (u-v)(u-v)(u-v) + a(u-v) + b \\
&= \left(u^2 - 2uv + v^2\right)(u-v) + au - av + b \\
&= u^3 - 2u^2v + uv^2 + 2uv^2 - v^3 + au - av + b \\
&= u^3 - v^3 - 3u^2v + 3uv^2 + au - av + b \\
&= \left(b - \left(v^3 - u^3\right)\right) + (u-v)(a - 3uv)
\end{aligned}
$$

Provided we use our freedom to choose u and v such that $a = 3uv$ and $b = v^3 - u^3$, then we can solve this equation. We shall look to solve these two equations simultaneously, meaning we shall try to find values of u and v that satisfy both equations. If we can do so, then we will have a complete solution. We start by looking at $a = 3uv$ and noting that this is equivalent to finding $v = \frac{a}{3u}$ when $u \neq 0$. We will check our final result for conditions under which $u = 0$, and find an approach to deal with those later. We can substitute this expression into $b = v^3 - u^3$ and find

$$
\begin{aligned}
b &= v^3 - u^3 \\
b &= \left(\frac{a}{3u}\right)^3 - u^3 \\
b &= \frac{a^3}{27u^3} - u^3 \\
27bu^3 &= a^3 - \left(u^3\right)^2 \\
\left(u^3\right)^2 + 27bu^3 - a^3 &= 0
\end{aligned}
$$

The choice to use $\left(u^3\right)^2$ in place of u^6 is deliberate. This reveals that the underlying structure here is one we can solve: this is quadratic in u^3. We can apply what we learned in the last lesson to find that

$$
u^3 = \frac{-9b \pm \sqrt{81b^2 + 12a^3}}{18}
$$

We can use this in the equation $b = v^3 - u^3$ to find that

$$
v^3 = b + u^3 = b + \frac{-9b \pm \sqrt{81b^2 + 12a^3}}{18} = \frac{9b \pm \sqrt{81b^2 + 12a^3}}{18}
$$

This gives us what we need to solve for the root x. Since $x = u - v$, the root can be found at

$$
x = \left(\sqrt[3]{\frac{-9b \pm \sqrt{81b^2 + 12a^3}}{18}}\right) - \left(\sqrt[3]{\frac{9b \pm \sqrt{81b^2 + 12a^3}}{18}}\right)
$$

Note that this gives us two possible roots. We must choose the same sign for both u and v when we have the $\pm$ choice, so there are only two possibilities. First we choose a sign for u, either the top or bottom, and then carry that choice through as we solve for v, and get matching signs. Then plug this answer into the original polynomial to verify that this is a root; if not, try the other sign. Once we have such a root, we'll need polynomial long division to extract any other roots. We will see that later.

2.2.1 The case when $u = 0$

If $u = 0$, then $u^3 = 0$. Thus,

$$\begin{aligned} 0 &= \frac{-9b \pm \sqrt{81b^2 + 12a^3}}{18} \\ 0 &= -9b \pm \sqrt{81b^2 + 12a^3} \\ 9b &= \pm\sqrt{81b^2 + 12a^3} \\ 81b^2 &= 81b^2 + 12a^3 \\ 12a^3 &= 0 \\ a &= 0 \end{aligned}$$

Thus, $u = 0$ if and only if $a = 0$. In this case, our above method for Case 2 doesn't apply. This will mean, however, that our original polynomial would be

$$P(x) = a_3x^3 + a_0$$

and we simply apply our methods from Case 1.

2.3 Case 3: The General Case

With the general case, we have two options. We can use the form

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

and solve for x, if we want to take the long, time consuming and masochistic option. Instead, we will take the second option, finding a way to transform any polynomial in this general form into a polynomial of the form found in case 2. With that transformation complete, we can solve the simplified version with the known process, and then transform our answer back into the form found here.

We are looking to solve

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

so we start by dividing by a_3 in order to simplify the process. By defining $p = \frac{a_0}{a_3}$, $q = \frac{a_1}{a_3}$ and $r = \frac{a_2}{a_3}$, which reduces our polynomial to

$$x^3 + rx^2 + qx + p = 0$$

Ideally, we would like to find a change of variables along the lines of $x = (y + k)$ which eliminates the quadratic term, i.e. after substituting for this variable, the coefficient of y^2 would be zero. Now we need only to substitute $x = y + k$ and solve for k, assuming this is possible.

$$\begin{aligned} x^3 + rx^2 + qx + p &= (y+k)^3 + r(y+k)^2 + q(y+k) + p \\ &= y^3 + 3y^2k + 3yk^2 + k^3 + ry^2 + 2rky + rk^2 + qy + kq + p \\ &= y^3 + (3k + r)y^2 + \left(3k^2 + 2rk + q\right)y + \left(k^3 + rk^2 + kq + p\right) \end{aligned}$$

To find our transformation, we let $3k+r = 0$ and solve for k, so that $k = -\frac{k}{3}$. Thus, our transformation is

$$x = y - \frac{r}{3}$$

transforming

$$x^3 + rx^2 + qx + p = 0$$

into

$$\begin{aligned} \left(y - \frac{r}{3}\right)^3 + r\left(y - \frac{r}{3}\right)^2 + q\left(y - \frac{r}{3}\right) + p &= 0 \\ y^3 + \left(q - \frac{6r^2}{9}\right)y + \left(p - \frac{rq}{3} - \frac{4r^3}{27}\right) &= 0 \end{aligned}$$

As complicated as those coefficients are, they are all known quantities which can be applied to case 2 to extract our first root. We are now left wondering only whether or not we can find others, and the answer is "yes."

3 Polynomial Long Division

Way back in lesson 14, we defined long division. We can attempt to do this again. After all, we can easily see how a polynomial of the form

$$P(x) = a_3(x - r_1)(x - r_2)(x - r_3)$$

would have the roots r_1, r_2 and r_3 and be easy to solve. If that's the case, then we can try to divide $ax^3 + bx^2 + cx + d$ by $x - r_1$ and reduce our cubic equation, we hope, to a quadratic equation. This would allow us to use any process above to get a specific root, and then extract that from the problem, turn it into a quadratic equation and continue as per last issue. We use the same logic that we used for regular division, as seen when dividing $x^3 + 3x^2 + 3x + 1$ by $x + 1$ in this example:

$$
\begin{array}{r|l}
 & \quad\quad\quad x^2 + 2x + 1 \\
\hline
x + 1 & x^3 + 3x^2 + 3x + 1 \\
 & -x^3 - x^2 \\
\hline
 & \quad 2x^2 + 3x \\
 & \quad -2x^2 - 2x \\
\hline
 & \quad\quad\quad x + 1 \\
 & \quad\quad\quad -x - 1 \\
\hline
 & \quad\quad\quad\quad 0
\end{array}
$$

If we divide $ax^3 + bx^2 + cx + d$ by $x - r_1$, leaving things in the general form, we get a dividend of $ax^2 + (b + ar_1)\, x + \left(ar_1^2 + br_1 + c\right)$ and a remainder of $ar_1^3 + br_1^2 + cr_1 + d = 0$, since we would already know that r_1 is a root of the cubic polynomial. While I would ideally be doing that inline and demonstrating it in its entirely, I haven't been able to format it via LaTeXin such a way that it renders properly. Instead, I will multiply $x - r_1$ into $ax^2 + (b + ar_1)\, x + \left(ar_1^2 + br_1 + c\right)$ and demonstrate that we get our original $ax^3 + bx^2 + cx + d$ back.

$$
\begin{aligned}
(x - r_1)\left(ax^2 + (b + ar_1)\, x + \left(ar_1^2 + br_1 + c\right)\right) &= ax^3 + bx^2 + ar_1x^2 + cx + br_1x + ar_1^2x \\
&\quad -ar_1x^2 - br_1x - ar_1^2x - cr_1 - br_1^2 - ar_1^3 \\
&= ax^3 + bx^2 + cx - cr_1 - br_1^2 - ar_1^3 \\
&= ax^3 + bx^2 + cx + d
\end{aligned}
$$

where we have used the fact that $ar_1^2 + br_1 + c + d = 0$ in the last step.

We now have a means to solve any cubic equation. First, we ensure that $a_2 = 0$ in our polynomial via variable transformation, then solve that one as per case 1 or 2 (depending upon whether or not $a_1 = 0$), and use polynomial long division with the result to extract a quadratic. Then solve this quadratic for the other two roots using the quadratic equation formula, which simplifies to

$$x = \frac{-a_2 - a_3r_1 \pm \sqrt{a_2^2 - 3a_3^2r_1^2 - 2a_3a_2r_1 - 4a_3a_1}}{2a_3}$$

and use each of the signs in the $\pm$ option to solve for the last two roots. It is possible, perhaps even likely, that $a_2^2 - 3a_3^2r_1^2 - 2a_3a_2r_1 - 4a_3a_1 < 0$, which causes difficulties so long as the axiom of inequality is in play.

4 Next Lesson

Next lesson, we examine quartic equations, and find a general means to solve those. After that, we will throw out the axiom of inequality and move towards the Fundamental Theorem of Algebra.