Math From Scratch Lesson 38: Solving Quartics

W. Blaine Dowler

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1 Defining Quartics

A quartic polynomial is one in which the order of the polynomial is 4, so it can be represented as

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

where $a, b, c, d, e \in \mathbb{Z}$ and $a \neq 0$.

To find the roots of this equation, we look for a general method to either

1. solve $ax^4 + bx^3 + cx^2 + dx + e = 0$ for all possible roots, or
2. extract a single root, allowing us to use polynomial long division to reduce the remaining problem to a cubic equation, which we can already solve.

We will ultimately take the second approach. Again, we will start with simpler cases.

2 Finding Quartic Roots

2.1 Case 1: $b = d = 0$

This is perhaps the simplest case to deal with. When $b = d = 0$, our polynomial reduces to

$$P(x) = ax^4 + cx^2 + e$$

With a substitution of $y = x^2$, this is reduced to

$$P(x) = ay^2 + cy + e$$

This is of quadratic form, so we can apply the techniques for quadratics to solve for y, and then use those results to solve for x. These are known as *biquadratic forms.*

2.2 Case 2: $b = 0$

We are going to solve problems of the type

$$0 = x^4 + cx^2 + dx + e$$

as our next special case. If $a \neq 1$, we can divide by a and produce a new set of coefficients to continue the program.

We start by trying to make it look as simple as possible by moving the cx^2 and dx terms to the other side of

$$x^4 + cx^2 + dx + e = 0$$

as

$$x^4 + e = -dx - cx^2$$

Now we use a technique similar to completing the square and add $2\sqrt{e}x^2$ to both sides of the equation:

$$x^4 + 2\sqrt{e}x^2 + e = -dx - cx^2 + 2\sqrt{e}x^2$$

so we can now write this as

$$\left(x^2 + \sqrt{e}\right)^2 = \left(2\sqrt{e} - c\right)x^2 - dx$$

Next comes the stroke of brilliance that I had to look up, since I was unable to come up with it on my own. The left hand side is a perfect binomial square in x, but the right hand side is not, as there are no constant terms. We can create such a constant term, though. We add an additional term to the bracket on the left hand side in the form of new variable y, and corresponding terms to the right hand side:

$$\left(x^2 + \sqrt{e} + y\right)^2 = \left(2\sqrt{e} - c\right)x^2 - dx + 2\sqrt{e}y + y^2 + 2x^2 y$$

As ugly as this looks, the left hand side is a perfect square. Thus, so is the right hand side. In the context of a quadratic

$$ax^2 + bx + c = 0$$

this would be a perfect square. In other words, we can rewrite it as

$$a\left(x + \frac{b}{2a}\right)^2 = 0 = ax^2 + bx + \frac{b^2}{4a}$$

which implies that $\frac{b^2}{4a} = c$. This means that

$$\begin{aligned} \frac{b^2}{4a} &= c \\ b^2 &= 4ac \\ b^2 - 4ac &= 0 \end{aligned}$$

This may seem familiar. It is the piece known as the *discriminant* of the quadratic equation. We will eventually show that every polynomial has a discriminant, and that such a discriminant is zero if and only if we have a repeated root to our polynomial. In our case, we want to force our chosen y to be of a form which ensures that the discriminant above is zero.

First, we rewrite the right hand side of our above expression to collect it as a quadratic in x:

$$\left(2\sqrt{e} - c\right)x^2 - dx + 2\sqrt{e}y + y^2 + 2x^2 y = \left(2\sqrt{e} - c + 2y\right)x^2 - dx + \left(2\sqrt{e}y + y^2\right)$$

Now we form the quadratic and set it equal to zero:

$$\begin{aligned} (-d)^2 - 4\left(2\sqrt{e} - c + 2y\right)\left(2\sqrt{e}y + y^2\right) &= 0 \\ d^2 - 4\left(4ey + 2\sqrt{e}y^2 - 2c\sqrt{e}y - cy^2 + 4\sqrt{e}y^2 + 2y^3\right) &= 0 \\ 8y^3 + \left(24\sqrt{e} - 4c\right)y^2 + \left(16e - 8\sqrt{e}\right)y - d^2 &= 0 \end{aligned}$$

This is now a cubic equation in y, allowing us to solve for y. With that solved, we can now substitute it back into

$$\left(x^2 + \sqrt{e} + y\right)^2 = \left(2\sqrt{e} - c\right) x^2 - dx + 2\sqrt{e}y + y^2 + 2x^2 y$$

and transform the right hand side into something we can factor more easily. We can then take the square root of both sides, transforming the entire equation into a quadratic that is easy to solve. Actually completing the details with the general form of y to find the general solution to x is remarkably cumbersome and not particularly illuminating, so the details will be omitted.

2.3 Case 3: The General Case

We will continue taking the second option for finding the general case, finding a way to reduce the general case of

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

into a form with no bx^3 term through a change of variables. As with cubic equations, we can manage this with a change of variables of the form $x = y + k$. This is equivalent to solving

$$a(y+k)^4 + b(y+k)^3 + c(y+k)^2 + d(y+k) + e = 0$$

with a particular form of k. We can find this k by looking specifically at the terms with y^3. Expanding this in full gives us

$$ay^4 + (4ak + b)\, y^3 + (\ldots)\, y^2 + (\ldots)\, y + (\ldots) = 0$$

where we have omitted the lengthy coefficients of lower order y terms. The focus is to solve for k:

$$\begin{aligned} 4ak + b &= 0 \\ 4ak &= -b \\ k &= -\frac{b}{4a} \end{aligned}$$

Thus, a substitution of $x = y - \frac{b}{4a}$ into

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

will reduce the quartic into a form as in case 2. This is the last step needed to solve any quartic polynomial.

3 Next Lesson

In our next lesson, we will discard the axiom of inequality and open thing up to far more possibilities.