Math From Scratch Lesson 39: Losing the Axiom of Inequality

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1 Motivation

We are finally prepared to lose the axiom of inequality, and move forward from there. Before doing that, we'll review the issues we have with using the real numbers exclusively, then take a good, hard look at the axioms we used for them and illustrate why we can make progress if we remove that axiom in particular.

2 The Analysis

2.1 The Axioms

The axioms we use for the real numbers are as follows:

1. **Closed under +** $\forall x, y \in \mathbb{R} x + y \in \mathbb{R}$
2. **Closed under** $\cdot$ $\forall x, y \in \mathbb{R} x \cdot y \in \mathbb{R}$
3. **Commutes under +** $\forall x, y \in \mathbb{R}, x + y = y + x$
4. **Commutes under** $\cdot$ $\forall x, y \in \mathbb{R}, x \cdot y = y \cdot x$
5. **Associative under +** $\forall x, y, z \in \mathbb{R}, x + (y + z) = (x + y) + z$
6. **Associative under** $\cdot$ $\forall x, y, z \in \mathbb{R}, x \cdot (y \cdot z) = (x \cdot y) \cdot z$
7. **Identity under +** $\exists 0 \in \mathbb{R} : \forall x \in \mathbb{R} x + 0 = x$
8. **Identity under** $\cdot$ $\exists 1 \in \mathbb{R} : \forall x \in \mathbb{R} x \cdot 1 = x$
9. **Inverses under +** $\forall x \in \mathbb{R} \exists - x : x + (-x) = 0$
10. **Inverses under** $\cdot$ $\forall x \neq 0 \in \mathbb{R} \exists x^{-1} : x \cdot x^{-1} = 1$
11. **Distributive Property** $\forall x, y, z \in \mathbb{R} x \cdot (y + z) = x \cdot y + x \cdot z$
12. **Completeness Axiom** This axiom states that every convergent sequence x_n where $x_n \in \mathbb{R} \forall n$ converges to a limit $L \in \mathbb{R}$. In other words, if the limit of a sequence of real numbers exists, then that limit is also a real number.
13. **Axiom of Inequality** $\forall x, y, z \in \mathbb{R}$ if $x < y$ and $y < z$ then $x < z$

2.2 The Limitations

The limitations we have using only these axioms are as follows:

1. We cannot calculate square roots (or any even roots) of negative numbers.

2. Due to the above problem, we are also unable to find roots for all polynomials.

These may not sound like terribly problematic limitations, largely because many public school systems never overcome them, but they are limitations nonetheless. As we are slowly but surely building a mathematical series using an axiomatic approach, we are under no obligation to accept limitations, *if and only if* we can concoct a consistent set of axioms that would eliminate those limitations. We shall see that we can overcome these limitations, if we remove one of the above axioms and insert a new one.

2.3 Evaluating Each Axiom

2.3.1 Closure under Addition

The closure axiom is needed to ensure that the system is consistent. We need to slightly modify the notation, since we do know that the real numbers are not going to get the job done. I'll mark the set with the symbol $\mathbb{C}$, because I'm either psychic or planning ahead.

$\forall x, y \in \mathbb{R} x + y \in \mathbb{C}$

2.3.2 Closure under Multiplication

Similarly, this becomes $\forall x, y \in \mathbb{R} x \cdot y \in \mathbb{C}$.

2.3.3 Commutes under Addition

Since we are working with this system to work through issues with polynomials, it is only natural that we want to keep the commutation axioms. $\forall x, y \in \mathbb{C}, x + y = y + x$

2.3.4 Commutes under Multiplication

Similarly, $\forall x, y \in \mathbb{C}, x \cdot y = y \cdot x$

2.3.5 Associative under Addition

We retain this for the same reasons we retained the closure and commutative properties: $\forall x, y, z \in \mathbb{C}, x + (y + z) = (x + y) + z$

2.3.6 Associative under Multiplication

Again, similarly, $\forall x, y, z \in \mathbb{C}, x \cdot (y \cdot z) = (x \cdot y) \cdot z$

2.3.7 Identity under Addition

For polynomials to retain their properties, we need to ensure that $\exists 0 \in \mathbb{C} : \forall x \in \mathbb{C} x + 0 = x$

2.3.8 Identity under Multiplication

Similarly, $\exists 1 \in \mathbb{C} : \forall x \in \mathbb{C} x \cdot 1 = x$

2.3.9 Inverses under Addition

We need to keep this one too: $\forall x \in \mathbb{C} \exists - x : x + (-x) = 0$

2.3.10 Inverses under Multiplication

$\forall x \neq 0 \in \mathbb{C} \exists x^{-1} : x \cdot x^{-1} = 1$

2.3.11 Distributive Property

$\forall x, y, z \in \mathbb{C} x \cdot (y + z) = x \cdot y + x \cdot z$

2.3.12 Completeness Axiom

This axiom states that every convergent sequence x_n where $x_n \in \mathbb{C} \forall n$ converges to a limit $L \in \mathbb{C}$. In other words, if the limit of a sequence of numbers in the set

$\mathbb{C}$ exists, then that limit is also a number is in the set $\mathbb{C}$. Without this axiom, we lose the ability to take square roots of arbitrary positive numbers, so we'd lose the ability to solve polynomials we already have solutions to. That's adding to our limitations, rather than removing them, so that's not a promising route. It's a path we may need to take, but for now, we'll call that plan B.

2.3.13 Axiom of Inequality

This is our last and best hope for getting through our problems. Thankfully, it will work and this is the option that we'll ultimately do.

The Axiom of Inequality was needed primarily for visualization and convenience. It allowed us to define new relations in the form of $<$, $>$, $\leq$ and $\geq$ and allowed us to create the notion of a number line. It also allowed us to define division when coupled with absolute values, so that we could use positive or negative divisors without concern. Thus, if we can find a way to extend the numbers in our set to allow for the square roots of negative numbers, while still maintaining a sensible definition of absolute values, then we can solve our problems without introducing new ones.

3 The New Axiom

We shall reject the Axiom of Inequality and instead institute the following new axiom:

- **Imaginary Axiom**: There exists a number i such that $i^2 = -1$. Formally, $\exists i \in \mathbb{C} : i^2 = -1$.

We can define the set of imaginary numbers $\mathbb{I}$ as the set of numbers of the form bi such that $b \in \mathbb{R}$. We can define the set of complex numbers $\mathbb{C}$ as the set of numbers of the form $a + bi$ such that $a, b \in \mathbb{R}$. All that remains is to determine if either or both of these sets is an algebraic field which is compatible with polynomials. If so, then we need to develop a definition of absolute value so that we can define the operation of division as well.

4 Next Lesson

In our next lesson, we will show that one of the sets $\mathbb{I}$ or $\mathbb{C}$ does form a field while the other does not, and will define the absolute value of the elements of both sets.