Math From Scratch Lesson 41: The Fundamental Theorem of Algebra

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1 Algebraically Closed Sets

We have discussed and defined polynomials before. It is possible to create polynomials with coefficients from any algebraic field based on a set S of any degree. We can then use these polynomials for a variety of studies. The set of all polynomials whose coefficients are drawn from this set is denoted $S[x]$. This allows us to explore another new concept.

A set of polynomials $S[x]$ is *algebraically closed* if every polynomial in $S[x]$ has all of its roots in the set S. For example, let us define the set $S = \mathbb{Z}_3$, which is the set of the integers modulo 3, as defined in volume 1 of this series. If every polynomial of every degree has its roots in this set, then it is algebraically closed. If we look only at polynomials of the form $x - a$ with $a \in S$, then every polynomial has a root in $S[x]$. If we allow any polynomial of any degree, as we must do to be properly algebraically closed, then we come to a different conclusion. If we look at all polynomials of degree 2, then we have the following form for all polynomials in $S[x]$: $ax^2 + bx + c$ where $a, b, c \in \mathbb{Z}_3$. The elements of $\mathbb{Z}_3$ are 0, 1 and 2, with the following addition and multiplication tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

There are 18 possible degree 2 polynomials in this set:

1. $x^2 + 0x + 0$
2. $x^2 + 0x + 1$
3. $x^2 + 1x + 0$
4. $x^2 + 1x + 1$
5. $x^2 + 0x + 2$
6. $x^2 + 2x + 0$
7. $x^2 + 1x + 2$
8. $x^2 + 2x + 1$
9. $x^2 + 2x + 2$
10. $2x^2 + 0x + 0$
11. $2x^2 + 0x + 1$
12. $2x^2 + 1x + 0$
13. $2x^2 + 1x + 1$
14. $2x^2 + 0x + 2$
15. $2x^2 + 2x + 0$
16. $2x^2 + 1x + 2$
17. $2x^2 + 2x + 1$
18. $2x^2 + 2x + 2$

We have 2 choices for a (as $a \neq 0$ or the polynomial would not be degree 2) but three choices each for b and c. Now let us examine all possibilities with roots. If a polynomial can be factored with roots, then it will be of the form $a\left(x+d\right)\left(x+e\right)$, where there are, again, 2 choices for a and three choices each for d and e. This sounds as though we'll have the same total of 18 results, but let us manually calculate them for this particular case.

1. $(x+0)(x+0) = x^2 + 0x + 0$
2. $(x+0)(x+1) = x^2 + 1x + 0$
3. $(x+1)(x+0) = x^2 + 1x + 0$
4. $(x+1)(x+1) = x^2 + 2x + 1$
5. $(x+0)(x+2) = x^2 + 2x + 0$
6. $(x+2)(x+0) = x^2 + 2x + 0$
7. $(x+1)(x+2) = x^2 + 0x + 2$
8. $(x+2)(x+1) = x^2 + 0x + 2$
9. $(x+2)(x+2) = x^2 + 1x + 1$
10. $2(x+0)(x+0) = 2x^2 + 0x + 0$
11. $2(x+0)(x+1) = 2x^2 + 2x + 0$
12. $2(x+1)(x+0) = 2x^2 + 2x + 0$
13. $2(x+1)(x+1) = 2x^2 + 1x + 2$
14. $2(x+0)(x+2) = 2x^2 + 1x + 0$
15. $2(x+2)(x+0) = 2x^2 + 1x + 0$
16. $2(x+1)(x+2) = 2x^2 + 0x + 1$
17. $2(x+2)(x+1) = 2x^2 + 0x + 1$
18. $2(x+2)(x+2) = 2x^2 + 2x + 2$

Notice that this list has duplications on the right, as a result of the commutative property. The polynomials created by $(x+0)(x+1)$ and $(x+1)(x+0)$ are identical polynomials. Of the 18 possible polynomials from our first list, 6 cannot be produced by constructing a polynomial with the roots listed. These six must not have roots, meaning that no value of x will cause the polynomial to equal zero. To verify this, we can evaluate each of these six polynomials for each of the three possible values of x, as seen in this table:

Polynomial	$x=0$	$x=1$	$x=2$
x^2+0x+1	1	2	2
x^2+1x+2	2	1	2
x^2+2x+2	2	2	1
$2x^2+0x+2$	2	1	1
$2x^2+2x+1$	1	2	1
$2x^2+1x+1$	1	1	2

These polynomials, which cannot be expressed as the product of simpler polynomials, are called *irreducible polynomials*.

Thus, $\mathbb{Z}_3[x]$ is not closed. If you examine the degree 2 polynomials based on $\mathbb{Z}_4[x]$, you will find that there are 48 possible polynomials, but only 19 of those can actually be factored.

While these examples are illustrative, they are only specific examples. There is, however, a generalization that can be made from this line of reasoning: $S[x]$ is *not* algebraically closed if S has a finite number of elements. For example, let us assume that S has a finite number of elements n. There are either $(n-1)\,n^{d-1}$ or n^d polynomials of degree d which can be constructed, with the former number representing the case where $0 \in S$ and the latter when it is not. When constructing a list of all possible reducible polynomials, you will have options that differ only by commuting the factors, and will therefore have duplicates every single time. The only way for every polynomial to have roots and for $S[x]$ to be algebraically closed is if there is an infinite number of possibilities, so that we do not "run out" of options.

2 If S is Infinite, is $S[x]$ Algebraically Closed?

We now see that finite sets S cannot generation algebraically closed sets $S[x]$. Is having an infinite S enough? No, it is not. This is a very simple proof to do: all we need to do is find even a single example of a set S which is infinite that has even one polynomial in $S[x]$ which does not have a root in S. For this, we use $S = \mathbb{Z}$, the set of all integers, and look at the polynomial $2x + 1$. The root of this polynomial can be found by starting with $2x + 1 = 0$ and solving for x.

$$\begin{aligned} 2x + 1 &= 0 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

As $-\frac{1}{2} \notin \mathbb{Z}$, we have proven that simply having an infinite S is not sufficient to make $S[x]$ algebraically closed. Let us now examine our other commonly used sets to see if they are any different.

- Rational numbers $\mathbb{Q}$: $x^2 - 2 = 0$ has roots at $x = \pm\sqrt{2}$, and is not algebraically closed.
- Irrational numbers $\overline{\mathbb{Q}}$: $\left(\sqrt{2}\right) x - 3\sqrt{2} = 0$ has a root at $x = 3$, and is not algebraically closed.

- Real numbers $\mathbb{R}$: $x^2 + 1 = 0$ has roots at $x = \pm i$, and is not algebraically closed.
- Imaginary numbers $\mathbb{I}$:$ix - i = 0$ has a root at $x = 1$, and is not algebraically closed.

This leaves only the complex numbers to examine. There is no immediate and obvious example of a polynomial in $\mathbb{C}[x]$ which has roots outside $\mathbb{C}$, so we will need to take the more rigorous and formal approach of proving algebraically whether or not $\mathbb{C}[x]$ is algebraically closed. In fact, $\mathbb{C}[x]$ is algebraically closed, although we won't be able to prove that until we establish some results from calculus and analysis, and for that, we need to get to graphing. Thus, the proof of this statement will have to wait.

3 Next Lesson

In our next lesson, we introduce vectors formally, and start working towards graphing and calculus.