Math From Scratch Lesson 42: n-tuples, Vectors and Vector Spaces

W. Blaine Dowler

March 23, 2014

Contents

1 Expanding Horizons

We have now seen complex numbers, which are named for the original meaning of "complex," or having multiple parts. That prepares us to talk in general about a broader scope of mathematical objects, where our variables such as x may refer to more than a simple number. We have used some of these previously in examples, such as matrices, but this is the first time we'll put them into formalisms that are used as the foundation for future results. With our notion of mathematical objects broadened in such a way, we can start to move towards n-tuples, vectors, matrices, one-forms, operators and other more advanced objects.

2 n-tuples

The first of the objects we will define is an n-tuple. This is the most general form of a class of objects which includes vectors, matrices and one-forms. The idea of an n-tuple is not particularly different from the idea of a set. Sets, defined way back in our second lesson, are collections of any objects and have no intrinsic order. An n-tuple is a collection of elements drawn from a single set which is organized in a particular order. For example, {Spider-Man, 5} and {5, Spider-Man} are identical sets. These elements cannot be used to populate a single n-tuple, as Spider-Man and 5 are not the same types of mathematical objects. 4 and 5 are the same types of objects, but the n-tuples $(4, 5)$ and $(5, 4)$ are distinct objects because the order of the entries matters. (Note that we use the round brackets () for n-tuples to make them appear distinct from the sets which use {} for their markers.) These examples are 2-tuples, as they contain 2 elements. $(1, 2, 3)$ is a 3-tuple, and $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ is an n-tuple.

3 Vectors and One-Forms

n-tuples are all well and good, but what can we do with them? Well, frankly, very little until we define operations that govern them. One type of n-tuple is a *vector*. The definition of a vector that we typically see first is that specific to its definition in high school physics classes: vectors are objects which include both magnitude and direction. This is a perfectly valid, but somewhat limited definition, designed for a specific application. We will use a more general definition, which is good, because we are still a long way from graphing and connecting these objects to geometry, thereby defining the term "direction" in a mathematical sense. Vectors will be denoted by an arrow above the letter, such as $\vec{u} = (u_1, u_2, u_3, \ldots, u_n)$. The elements of a vector may be real numbers, complex numbers, or any other set of numbers such that 1 and 0 are elements of the set and the set is a ring (or field, or...) under addition and multiplication.

First, we define a few operations and standard objects:

1. Zero vector: $\vec{0} = (0, 0, 0, 0, \ldots, 0)$
2. Negative vector: if $\vec{u} = (u_1, u_2, u_3, \ldots, u_n)$, then $-\vec{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$.
3. Addition: $\vec{u} = (u_1, u_2, u_3, \ldots, u_n) + \vec{v} = (v_1, v_2, v_3, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n)$
4. Subtraction: $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

5. Scalar multiplication: $a\vec{u} = (au_1, au_2, au_3, \ldots, au_n)$ where a is a member of the same set as the various u_i.

If the above definitions hold, then the objects we are dealing with are either vectors or one-forms. The distinction between vectors and one-forms will not come for quite some time, as it depends on products of vectors, which we have yet to define. One important distinction is in the way they are written outside of a comma separated list. Vectors, when written without commas, are written as columns:

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}$$

Conversely, one-forms are denoted with a tilde above instead of an arrow ($\tilde{u} = (u_1, u_2, u_3, \ldots, u_n)$) and are written as rows:

$$\tilde{u} = \begin{pmatrix} u_1 & u_2 & u_3 & \cdots & u_n \end{pmatrix}$$

Each vector has an associated one-form, and vice versa.

Our vectors $\vec{u} = (u_1, u_2, u_3, \ldots, u_n), u_i \in S$ can form a vector space V if the following axioms are true:

1. $\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V$
2. $\vec{u} \in V, a \in S \Rightarrow a\vec{u} \in V$
3. $\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} = \vec{v} + \vec{u}$
4. $\vec{u}, \vec{v}, \vec{w} \in V \Rightarrow \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
5. $a, b \in S, \vec{u} \in V \Rightarrow a(b\vec{u}) = (ab)\vec{u}$.
6. $\vec{u} \in V \Rightarrow \vec{0} + \vec{u} = \vec{u}$
7. $\vec{u} \in V \Rightarrow 1\vec{u} = \vec{u}$
8. $\vec{u} \in V \Rightarrow -\vec{u} \in V, \vec{u} + (-\vec{u}) = \vec{0}$
9. $a \in S, \vec{u}, \vec{v} \in V \Rightarrow a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
10. $a, b \in S, \vec{u} \in V \Rightarrow (a + b)\vec{u} = a\vec{u} + b\vec{u}$

Note that rules 1 and 2 are closure axioms, 3 is a commutative axiom, 4 and 5 are associativity axioms, 6 and 7 are identity axioms, 8 is an inverse axiom and 9 and 10 are distributive property axioms. Note that none of these involve multiplying vectors by other vectors.

4 Inner Products and Inner Product Spaces

There are multiple definitions of vector products available which we will (eventually) see, but the one that is the most useful now is the inner product, denoted $\langle \vec{u}, \vec{v} \rangle$. The inner product is not a single product, but rather an entire class of products which satisfy the following properties, assuming $\vec{u}, \vec{v}, \vec{w} \in V$ and $a, u_i, v_i, w_i \in S$:

1. $\langle \vec{u}, \vec{v} \rangle \in S$
2. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle^*$
3. $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$
4. $\langle a\vec{u}, \vec{v} \rangle = a \langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, a^* \vec{v} \rangle$
5. $\langle \vec{u}, \vec{u} \rangle > 0 \forall \vec{u} \neq \vec{0}$

Note that rule 2 reduces to symmetry if S is not the complex number field (or other similar fields we have yet to define) as the complex conjugate will be the same as the original number. Note also that $\left\langle \vec{0}, \vec{0} \right\rangle = 0$ is not an explicit requirement here. That's because it is an *implicit* requirement; we simply apply property 3 to find that

$$\left\langle \vec{u}, \vec{0} \right\rangle = \left\langle \vec{u}, \vec{0} + \vec{0} \right\rangle = \left\langle \vec{u}, \vec{0} \right\rangle + \left\langle \vec{u}, \vec{0} \right\rangle = 2 \left\langle \vec{u}, \vec{0} \right\rangle$$

Thus, $\left\langle \vec{u}, \vec{0} \right\rangle = 2 \left\langle \vec{u}, \vec{0} \right\rangle$, or $\left\langle \vec{u}, \vec{0} \right\rangle = 0$.

We will conclude this month with an example of the most common inner product: the dot product. The dot product of two vectors, also known as the scalar product, is defined as follows:

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i^*$$

We now verify that this is, indeed, an inner product. To do this in the most general form, we will use $S = \mathbb{C}$. In the event that $S = \mathbb{R}$, all of our results involving complex conjugates reduce to more simple cases. Let us test these results explicitly.

4.1 $\langle \vec{u}, \vec{v} \rangle \in S$

For $\vec{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, v_3, \ldots, v_n)$, we have

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i^* \in \mathbb{C}$$

by virtue of the fact that $\mathbb{C}$ is closed under conjugation, addition and multiplication.

4.2 $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle^*$

Let $u_j = a_j + ib_j$ and $v_j = c_j + id_j$, where we have replaced the index variable i with j to avoid confusion between the index and $\sqrt{-1}$.

$$\begin{aligned}
\langle \vec{u}, \vec{v} \rangle &= \sum_{j=1}^{n} u_j^* v_j \\
&= \sum_{j=1}^{n} (a_j + ib_j)(c_j + id_j)^* \\
&= \sum_{j=1}^{n} (a_j + ib_j)(c_j - id_j) \\
&= \sum_{j=1}^{n} a_j c_j + ib_j c_j - ia_j d_j + b_j d_j \\
&= \sum_{j=1}^{n} (a_j c_j + ib_j c_j - ia_j d_j + b_j d_j)^* \\
&= \sum_{j=1}^{n} ((c_j + id_j)(a_j - ib_j))^* \\
&= \sum_{j=1}^{n} \left((c_j + id_j)(a_j + ib_j)^*\right)^* \\
&= \langle \vec{v}, \vec{u} \rangle^*
\end{aligned}$$

So, the second condition is satisfied for $\mathbb{C}$. For $\mathbb{R}$, simply set $b_j = d_j = 0$ and get the same result.

4.3 $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$

Use the same definitions as above, and add $w_j = e_j + i f_j$.

$$\begin{aligned}
\langle \vec{u} + \vec{w}, \vec{v} \rangle &= \sum_{j=1}^{n} (u_j + w_j)\, v_j^* \\
&= \sum_{j=1}^{n} u_j v_j^* + w_j {v_j}^* \\
&= \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle
\end{aligned}$$

4.4 $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, k^* \vec{v} \rangle$

If $u_j = a_j + i b_j$, $v_j = c_j + i d_j$ and $k = m + in$, then

$$\begin{aligned}
\langle k\vec{u}, \vec{v} \rangle &= \sum_{j=1}^{n} (m + in)\, u_j {v_j}^* \\
&= (m + in) \sum_{j=1}^{n} u_j v_j^* \\
&= k \langle \vec{u}, \vec{v} \rangle \\
&= \sum_{j=1}^{n} u_j k v_j^* \\
&= \sum_{j=1}^{n} u_j \left(k^* v_j\right)^* \\
&= \langle \vec{u}, k^* \vec{v} \rangle
\end{aligned}$$

where we have used the fact that $(k^*)^* = k$.

4.5 $\langle \vec{u}, \vec{u} \rangle > 0 \forall \vec{u} \neq \vec{0}$

$$\begin{aligned}
\langle \vec{u}, \vec{u} \rangle &= \sum_{j=1}^{n} u_j u_j^* \\
&= \sum_{j=1}^{n} |u_j|^2 \\
&\geq 0
\end{aligned}$$

From our previous definition, we see that $|u_j|^2 = 0 \Leftrightarrow u_j = 0$, so this property holds. That is the fifth and final property, so the dot or scalar product is an inner product.

5 Next Lesson

Now that we are armed with inner products and inner product spaces, we can venture towards graphs and graphing.