Math From Scratch Lesson 44: Solving Polynomials with Matrices

W. Blaine Dowler

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Contents

1 Review: Polynomial Roots

We begin with a quick review of polynomial roots. In particular, let us examine the simple polynomials $x - 3$ and $x^2 - 5x + 6$. The roots of these polynomials in the real or complex numbers can be found by setting the polynomial equal to 0 solving for x. For the first, the solution is $x = 3$, as $(3) - 3 = 0$, while the solutions to the second are 2 and 3, as $(2)^2 - 5(2) + 6 = 4 - 10 + 6 = 0$ and $(3)^2 - 5(3) + 6 = 9 - 15 + 6 = 0$. Notice that you must solve the equation with 2 and 3 individually in the latter case, as $(2)^2 - 5(3) + 6 = 4 - 15 + 6 = -5 \neq 0$ and $(3)^2 - 5(2) + 6 = 9 - 10 + 6 = 5 \neq 0$.

2 Linear Polynomial Matrix Roots

To solve $X - 3 = 0$ for matrix values of X, we must define $x - 3$. In the algebras we're used to, we can always multiply any term by the multiplicative identity for that algebra without impacting the truth of the equation. For example, if $x - 3 = 0$ then $X - 3 \cdot 1 = 0$ as well. Applying that logic here, we can multiply by the identity matrix I, so we would have $X - 3 \cdot I = 0$. Let us see if we can find a specific X which solves this equation in matrix form. At this stage, we'll just be using intelligent guessing to find these solutions, although we will reach the stage where we can derive them explicitly.

In the case of $x - 3 = 0$, we know that $x = 3$ is the traditional solution, so we will try matrices that feature the value 3 prominently. We start with

$$X = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

and give that a try. With this value, $X - 3$ becomes

$$X = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is not a solution to $X - 3 = 0$. If we look at this solution, we can see that

$$X = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

is a likely solution, which is easily verified as

$$X = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since matrix addition and subtraction operations happen one element at a time, and we have no matrix multiplication, we can easily use existing methods to check the uniqueness of this solution by solving for the general case.

Let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Thus, $X - 3$ works out to

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a-3 & b \\ c & d-3 \end{pmatrix}$$

which is only a solution if $a - 3 = b = c = d - 3 = 0$, which is only true if $a = d = 3$ and $b = c = 0$. That is exactly the matrix we found, so the solution is unique. Our linear algebraic polynomial as linear algebraic conditions on each of its matrix element solutions.

3 Quadratic Polynomial Matrix Roots

Let us now examine $X^2 - 5X + 6$ in a similar fashion. The traditional roots are 2 and 3, so let us try the above methods to solve the matrix. In the linear case, the normal algebra had a single solution, and the matrix algebra version also had a single solution. Let us see then if this quadratic has exactly two matrix solutions as well.

The first solution we try is

$$X = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

as it was successful in the previous attempt.

$$\begin{aligned} X^2 - 5X + 6 &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - 5\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

This solution clearly works. Now we try the comparable

$$X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

as 2 and 3 were the traditional solutions. This gives us:

$$\begin{aligned} X^2 - 5X + 6 &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - 5\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

which also works. Do we only have these two solutions? Let us try what is perhaps the simplest alternative:

$$X = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

which gives us

$$\begin{aligned} X^2 - 5X + 6 &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - 5\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

which is also a solution. Clearly, there are more than two matrix solutions. It is not difficult to show that

$$X = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

is also a solution. That brings us up to four solutions. Can we find a fifth? Indeed, we can. A "guess and test" method may get us there, or I can apply knowledge and methods we haven't developed yet to arrive at

$$X = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}$$

Explicitly trying this, we get

$$\begin{aligned} X^2 - 5X + 6 &= \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix} - 5 \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix} + 6 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{13}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{13}{2} \end{pmatrix} - \begin{pmatrix} \frac{25}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{25}{2} \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

so this is also a solution. If we use the general form

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we can see that any matrix subject to the conditions

$$X^2 - 5X + 6 = \begin{pmatrix} a^2 + a + bc & (a + d + 1)\, b \\ (a + d + 1)\, c & bc + d + d^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If $b = c = 0$, then the restraints reduce to $a^2 + a = d^2 + d = 0$, so that each of a and d can be either 0 or -1. If at least one of b and c are not 0, then we have more complicated conditions, starting with $a + d + 1 = 0$. We will see that there are infinitely many possibilities to solve for a matrix X in a polynomial equation of degree 2 or higher, but that the roots of the polynomial over complex values of x are very tightly connected to the matrices that serve as solutions to the matrix equation. How do the numbers 2 and 3 relate to the matrix

$$X = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}$$

in any way? Well, to find out, we multiply by a couple of carefully selected matrices. Let

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then we find

$$XA = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} + \frac{1}{2} \\ \frac{1}{2} + \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3A$$

and

$$XB = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \frac{1}{2} \\ -\frac{5}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2B$$

We shall see that we can develop a methodology for determining what these column matrices are, and find specific and important applications of these values.

4 Next Lesson

In our next lesson, we will look at vector bases and operators.