Math From Scratch Lesson 45: Basis Vectors

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1 Basis Vectors

We have seen that vectors have different components. They can be added, and they can be multiplied, at least by scalars. For example,

$$4\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

The question is simple: can we find a "recipe" that allows us to combine any possible vector out of a specific combination of vectors? Well, the answer is "yes," if we choose carefully.

For example, let

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

be a vector with N components. The vectors defined by

$$\begin{aligned} \vec{e}_1 &= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \vec{e}_2 &= \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \\ &\vdots \\ \vec{e}_N &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

In other words, the ith component of $\vec{e}_i = 1$, while every other component is 0. Then,

$$\vec{a} = \sum_{i=1}^{N} a_i \vec{e}_i$$

Thus, there is a "trivial" way to create any possible vector $\vec{a}$ using the set $\vec{e}_i$ of basis vectors. Note that, if you have N components in your vectors then you must have N vectors $\vec{e}_i$ in this formulation. Is that true in general? Can we have more or less vectors in the set and still create a basis of vectors to create any vector from? Can we make a set of basis vectors in other ways?

1.1 Multiplicity of Bases

Let us explore some of these questions with simple examples. For example, it is not difficult to show that

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Is there another way to build

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

with different vectors? For example, can we use

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\vec{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

instead?

This would mean we'd have

$$\vec{v} = c\vec{u}_1 + d\vec{u}_2$$

for some values of c and d. If we can find a way to calculate c and d, based solely on the values of the components of $\vec{v}$, a and b. With our definitions of $\vec{u}_1$ and $\vec{u}_2$, we would need to solve $c+d=a$ and $c-d=b$ in such a way as to eliminate dependence of c on d and vice versa; we would need to have a formulation in which c depends solely on a and b, and d depends solely on a and b. Looking at the second equation, we find that $c = b + d$. If we substitute this expression into the first equation, we find that $(b + d) + d = a$, or

$$d = \frac{a-b}{2}$$

Rearranging our first equation into $c = a - d$ and substituting the above gives

$$c = a - \frac{a-b}{2} = \frac{2a-a+b}{2} = \frac{a+b}{2}$$

Thus, we have found the combination of coefficients which will allow us to express any vector $\vec{v}$ in terms of $\vec{u}_1$ and $\vec{u}_2$. Can this be done with arbitrary basis vectors?

Let us try using arbitrary basis vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

instead. Can we use these to build an arbitrary vector

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

in a similar manner?

If we start with

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = e\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + f\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

we find the conditions are

$$e = \frac{ay_2 - by_1}{x_1y_2 - x_2y_1}, f = \frac{bx_1 - ax_2}{x_1y_2 - x_2y_1}$$

So, we cannot have just any solution. We can only find a solution such that $x_1y_2 - x_2y_1 \neq 0$, or $x_1y_2 \neq x_2y_1$. Thus, so long as neither vector is $\vec{0}$, we'll be able to arbitrarily choose one of our vectors, and still have much (but not total) freedom to choose the last component. What happens if one vector is a multiple of the other? In other words, what if $y_1 = ax_1$ and $y_2 = ax_2$? This is strictly forbidden: this is equivalent to saying that $ax_1x_2 \neq ax_2x_1$, which is an impossibility. Thus, we have one situation in which we cannot have a basis to assemble any vector: if $\vec{y} = a\vec{x}$. Are there any other situations in which we cannot have a solution? There are two cases to examine:

1.1.1 Case 1: $x_1 = 0$

If $x_1 = 0$, then we must have $x_2 \neq 0$, or $x_1y_2 - x_2y_1 = 0$. This leaves us with the following result:

$$\begin{aligned} x_1y_2 &\neq x_2y_1 \\ 0 &\neq x_2y_1 \\ 0 &\neq y_1 \end{aligned}$$

where the last step takes advantage of the fact that $x_2 \neq 0$. So, if the system *doesn't* work, then the above equation is *false*, and $y_1 = 0$. This means that $y_2 = ax_2$, since any two non-zero numbers are related by some scalar multiple. We can also say that $x_1 = ay_1$ as both sides of the equation are 0, so this is still our above condition: $\vec{y} = a\vec{x}$.

1.1.2 Case 2: $x_1 \neq 0$

If $x_1 \neq 0$, then we can say that $y_1 = ax_1$, where a is allowed to be 0. The condition for a successful set of basis vectors is now $x_1y_2 - x_2y_1 = x_1y_2 - ax_1x_2 \neq 0$, which can now be solved for y_2 as follows:

$$\begin{aligned} x_1y_2 - ax_1x_2 &\neq 0 \\ x_1y_2 &\neq ax_1x_2 \\ y_2 &\neq ax_2 \end{aligned}$$

This is the same condition. Thus, in a vector space where vectors have only two components, any non-zero vectors which do not satisfy the condition $\vec{y} = a\vec{x}$ can be used as a basis for the vector space. What happens in higher dimensions?

1.1.3 The 3 dimensional case

Let us try to create

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$$

by using the vectors

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \vec{e}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

This gives us the conditions:

$$v_1 = a + c \quad v_2 = a + b + 2c \quad v_3 = b + c$$

The first and third conditions give us ways to express both a and b in terms of c and the components of $\vec{v}$. Substitution gives us:

$$\begin{aligned} v_2 &= (v_1 - c) + (v_3 - c) + 2c \\ &= v_1 + v_3 + 2c - 2c \\ &= v_1 + v_3 \end{aligned}$$

This does not, in any way, produce a means to determine a, b or c, as they are no longer involved in the equation. Instead, is places restrictions on v_2 in terms of v_1 and v_3. Thus, our arbitrary vector $\vec{v}$ cannot be arbitrary, leaving us with a logical contradiction. We do not have $\vec{e}_1 = d\vec{e}_2$, $\vec{e}_1 = f\vec{e}_3$ or $\vec{e}_2 = g\vec{e}_3$, so the condition we found in the two dimensional case wasn't strong enough.

By inspection, we can see that $\vec{e}_3 = \vec{e}_1 + \vec{e}_2$. This gives us a hint as to the condition we need: it is known as *linear independence*. A set of N vectors is *linearly independent* if and only if the only solution to

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 + \ldots + a_N\vec{v}_N = 0$$

is

$$a_1 = a_2 = a_3 = a_4 = \ldots = a_N = 0$$

To verify this, we must first discuss the span of a set of vectors, and determine the exact number of solutions to a given set of linear equations. This will lead us into operators.

2 Next Lesson

Next, we examine the number of solutions possible for m equations with n unknowns, and look at how that impacts linear independence and vector bases. This will also give us a formal definition of a vector basis which we can use to replace our current, informal/intuitive definition.