How To Calculate Square Roots By Hand

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Abstract
This document will outline how to calculate square roots by hand, and then continue to explain why the method works.

1 The Algorithm

The method for calculating square roots by hand is a recursive algorithm. In other words, you come up with a reasonable first estimate ($x_1$), apply the same formula repeatedly, and turn your first estimate into a better and better estimate each time.

The formula for estimating the square root of a number $a$ is the following:

$$x_{n+1} = x_n + \frac{a}{x_n}$$

Let us apply this formula to find the square root of 10.\(^1\) We’ll take 3 as our reasonable first guess $x_1$.

$$x_2 = \frac{x_1 + \frac{a}{x_1}}{2}$$

$$x_2 = \frac{3 + \frac{10}{3}}{2}$$

$$x_2 = \frac{19}{6} = 3.16$$

In one iteration (application) of the recursive formula, we’ve gotten two decimal places closer to the answer. After two iterations, we have:

$$x_3 = \frac{x_2 + \frac{10}{x_2}}{2}$$

$$x_3 = \frac{19}{6} + \frac{10}{\left( \frac{19}{6} \right)}$$

$$x_3 = \frac{721}{228} = 3.16228070175\ldots$$

\(^1\)The accepted value is approximately 3.162277660168.
This got us another two decimal places closer to the answer. One more iteration\(^2\) is accurate to 11 decimal places. In fact, with each iteration, you not only get closer to the correct answer, but you start to approach that answer more rapidly. As these things go, this process is actually very rapid. This is the method most calculators actually use to calculate a square root. It can be reapplied as often as needed to achieve the desired accuracy. There is one major limitation, though: it will never arrive at the exact value of a square root unless your first guess is the exact value. For example, if your first estimate for the square root of 9 is 3, it will return 3 with each iteration. If your first estimate for the square root of 9 is 2, then you will get numbers that get closer and closer to 3 with each iteration, but none will be exactly 3.

\[ x = \sqrt{a} \]

\[ x_1 = x \]

\[ a < x < \frac{a}{x_1} \]

\[ x_2 = \frac{x_1 + \frac{a}{x_1}}{2} \]

At this point, it is generally reasonable to accept that the method continually works. The contraction mapping approach is the one that proves \(x_{n+1}\) is actually closer to the correct value than \(x_n\). (In some cases, the logical arguments like this one keep you in the general neighbourhood of the correct answer with each iteration, but no real long term progress is made.) Those readers who require more formal approaches are invited back to Bureau 42 in July 2013 for the summer school term which leads up to this result.

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\(^2\) If, instead, you assume that \(x_1 > x\), you will find the endpoints of the three part inequality reversed. If you follow the logic carefully, you’ll see that the important point is that \(x\) lies between \(x_1\) and \(\frac{a}{x_1}\), but doesn’t depend upon the initial endpoint involved.