# How To Calculate Square Roots By Hand

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**Abstract**

This document will outline how to calculate square roots by hand, and then continue to explain why the method works.

## 1 The Algorithm

The method for calculating square roots by hand is a *recursive algorithm*. In other words, you come up with a reasonable first estimate ($x_1$), apply the same formula repeatedly, and turn your first estimate into a better and better estimate each time.

The formula for estimating the square root of a number $a$ is the following:

$$x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2}$$

Let us apply this formula to find the square root of 10.[1] We'll take 3 as our reasonable first guess $x_1$.

$$\begin{aligned} x_2 &= \frac{x_1 + \frac{a}{x_1}}{2} \\ x_2 &= \frac{3 + \frac{10}{3}}{2} \\ x_2 &= \frac{19}{6} = 3.1\overline{6} \end{aligned}$$

In one iteration (application) of the recursive formula, we've gotten two decimal places closer to the answer. After two iterations, we have:

$$\begin{aligned} x_3 &= \frac{x_2 + \frac{10}{x_2}}{2} \\ x_3 &= \frac{\frac{19}{6} + \frac{10}{\left(\frac{19}{6}\right)}}{2} \\ x_3 &= \frac{721}{228} = 3.16228070175\ldots \end{aligned}$$

[1] The accepted value is approximately 3.162277660168.

This got us another two decimal places closer to the answer. One more iteration[2] is accurate to 11 decimals places. In fact, with each iteration, you not only get closer to the correct answer, but you start to approach that answer more rapidly. As these things go, this process is actually very rapid. This is the method most calculators actually use to calculate a square root. It can be reapplied as often as needed to achieve the desired accuracy. There is one major limitation, though: it will never arrive at the exact value of a square root unless your first guess is the exact value. For example, if your first estimate for the square root of 9 is 3, it will return 3 with each iteration. If your first estimate for the square root of 9 is 2, then you will get numbers that get closer and closer to 3 with each iteration, but none will be exactly 3.

# 2 The Explanation

The only formal mathematical proof of this that I am aware of requires measures of contraction maps, which not only prove that the algorithm converges, but can actually be used to measure how rapidly that convergence takes place. That is well beyond the scope of a teaching tidbit. Instead, I will use a less formal but compelling argument to illustrate the point.

Let's say that $x = \sqrt{a}$ is the exact value of the (positive) square root of $a$. Let's say our first estimate is $x_1 \neq x$. So, that means that $x_1 y = a$ for some value of $y$. The value of $y$ that is of interest, then, is $\frac{a}{x_1}$. In the case where $x_1 < x$, we can confidently say that $\frac{a}{x_1} > x$. We can say this because of properties of inequalities: if instead $x_1 < x$ and $\frac{a}{x_1} < x$, then $x_1 \frac{a}{x_1} < x^2 < a$, but algebraic cancelations reduce the far left to $a$, leaving us with $a < a$, which is impossible in the real number system.

So, we have established that $x_1 < x < \frac{a}{x_1}$, given the assumption $x_1 < x$.[3] So, the answer we're looking for is somewhere between $x_1$ and $\frac{a}{x_1}$. A reasonable second estimate, then, would be some number that is guaranteed to fall between $x_1$ and $\frac{a}{x_1}$. The simplest number that is guaranteed to be between the two of them is the mean average of the two:

$$x_2 = \frac{x_1 + \frac{a}{x_1}}{2}$$

At this point, it is generally reasonable to accept that the method continually works. The contraction mapping approach is the one that proves $x_{n+1}$ is actually closer to the correct value than $x_n$. (In some cases, the logical arguments like this one keep you in the general neighbourhood of the correct answer with each iteration, but no real long term progress is made.) Those readers who require more formal approaches are invited back to Bureau 42 in July 2013 for the summer school term which leads up to this result.

[2] $\frac{1039681}{328776}$

[3] If, instead, you assume that $x_1 > x$, you will find the endpoints of the three part inequality reversed. If you follow the logic carefully, you'll see that the important point is that $x$ lies between $x_1$ and $\frac{a}{x_1}$, but doesn't depend upon the initial endpoint involved.