Proving the Pythagorean Theorem

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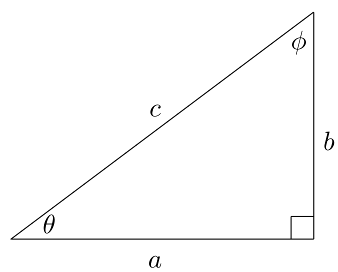
Abstract

Most people are familiar with the formula $a^2 + b^2 = c^2$. However, in most cases, this was presented in a classroom as an absolute with no attempt at a proof or derivation. This will now be derived here.

Contents

1 The Triangle

The Pythagorean Theorem applies only to right angled triangles; i.e. triangles in which one of the three interior angles has a measure of 90° or $\frac{\pi}{2}$ radians. The longest side, which is opposite this right angle, has a length c, while the other two sides have lengths a and b.[1] We label the two angles which are not right angles θ and ϕ as in the following diagram:



Note that, as usual, a box has been used in the corner to denote the right angle. With this notation, Pythagoras first proved that $a^2 + b^2 = c^2$.

2 The Preliminaries

There are a few prerequisite facts that need to be laid out in order to follow this proof. They are as follows:

- The sum of the three interior angles of any triangle (which has been drawn on a flat surface) is a constant equal to two right angles in whatever your preferred angular measure happens to be. (i.e. 180°, π radians, etc.)
- If you measure the "angle" of a straight line, as though it were actually two lines connected back to back, this angle is also equal to two right angles.
- The area of a square is the length of one side multiplied by itself.[2]
- The area of a triangle is $\frac{1}{2}bh$, where the base b is one side, and the height h is perpendicular distance between the base and the corner opposite the base. In a right angled triangle, labeled with sides a, b and c as above, this area reduces to $\frac{1}{2}ab$.

[1] While some individual teachers insist on the convention that a be the shortest side, there is no mathematical reason to make this a demand. It is purely a matter of aesthetics.

[2] This is exactly why raising x to exponent 2, x^2, is referred to as "squaring x."

- When squaring a binomial such as $a+b$, one gets "cross terms" involving the product of a and b. Written out explicitly,

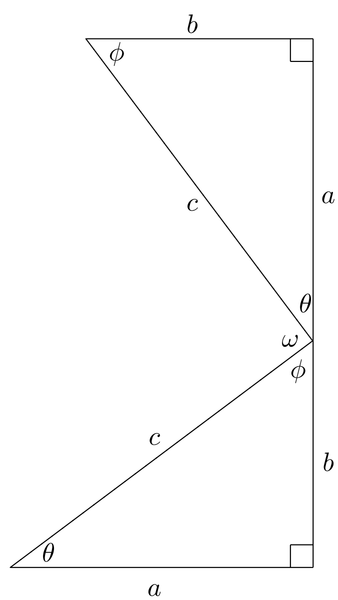
  $$(a+b)^2 = (a+b)(a+b) = a(a+b)+b(a+b) = a^2+ab+ba+b^2 = a^2+b^2+2ab$$

  where the $2ab$ cross term has been written last in defiance of convention for reasons that will soon be clear.

3 The Construction

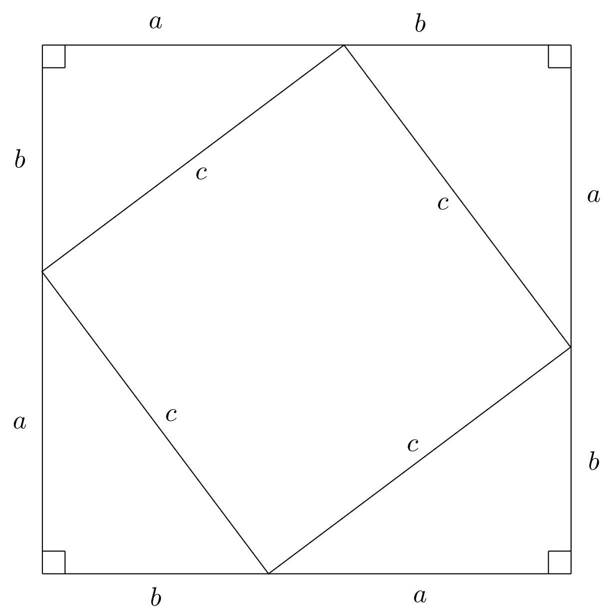
The following construction is far from the only way to derive and prove the Pythagorean Theorem. It is not even the one which Pythagoras himself used. (When he did his work, algebra was not a commonly accepted mathematical tool among his peers, and geometry was greatly preferred. Algebra will be used here.)

The construction begins by creating an exact copy of our triangle, and attaching it to the original triangle in a particular way, such that side b in the original is parallel to side a of the copy, and such that the copy has been rotated by a right angle counter clockwise before connecting the corners. The result looks like this:



Let us carefully examine the angles within this construction, including the newly labeled angle ω. As all three angles within our original triangle add up to two right angles, and as one corner is a right angle already, we can conclude that θ and ϕ must, together, also add up to a right angle. If we look at the corner where our two triangles meet, we see that (by design) we have a straight line running along our original side b and our new side a. Thus, on the left side of that joining, the angles θ, ϕ and ω must add up to two right angles. We know that θ and ϕ alone add up to a right angle, so we must conclude that ω alone, which is the angle between the two sides of length c, is also a right angle.

We duplicate our triangle twice more, and put all four triangles together in a similar manner to form this final construction:



4 The Proof

We are now in a position to derive and prove the Pythagorean theorem. Our above construction can be viewed in two different ways. On one hand, it is a giant square, with sides of length $a + b$. On the other hand, it is a smaller square[3] with sides length c surrounded by four right angle triangles with short sides a and b. We now calculate the area of this shape in two different ways. Treating the construction as a giant square gives us an area A equal to

$$A = (a + b)^2 = a^2 + b^2 + 2ab$$

[3] We know this is a square, as we have shown that the angle between the sides of length c, originally labeled ω, is actually a right angle.

while treating it as a small square surrounded by triangles gives us an area equal to

$$A = c^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 2ab$$

Now, as this is the same shape represented in two ways, we can equate the expressions for area and find ourselves left with

$$a^2 + b^2 + 2ab = c^2 + 2ab$$

or

$$a^2 + b^2 = c^2$$

after we cancel the common $2ab$. This is, indeed, the Pythagorean Theorem that is often taught by rote. We now have one way to prove that it is, indeed, true.