

# Turning Repeating Decimals Into Fractions: Algorithm and Reasoning

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## Abstract

This document details exactly how to convert any repeating decimal into its fractional form, and explains exactly why the method works.

## 1 The Algorithm

First, some notation. The overline notation will be used to represent the repeating portion of a decimal. For example,

$$12.343434343434\dots = 12.\overline{34}$$

Now, the algorithm for turning these repeating decimals into fractions is fairly straightforward. The explanation for it requires math generally left for High School, which will come later. Elementary school students can learn how to convert, even if they can't immediately be shown why it works.

When the repeating digits start right at the decimal, such as with  $0.\overline{3}$ , then we start by counting the number of digits that repeat. In  $0.\overline{3}$ , only one digit (the 3) repeats. In  $0.\overline{34}$ , two digits repeat (the 3 and 4.) We write the fractions by writing the repeating digit(s) over a matching number of 9s, remembering to reduce the fraction into lowest terms afterwards. So, for example,

$$\begin{aligned}0.\overline{3} &= \frac{3}{9} = \frac{1}{3} \\0.\overline{34} &= \frac{34}{99} \\0.\overline{123456789} &= \frac{123456789}{999999999} = \frac{13717421}{111111111} \\0.\overline{012} &= \frac{12}{999} = \frac{4}{333}\end{aligned}$$

Notice in the fourth example that we have three 9s in the denominator instead of just two. The 0 that repeats counts.

Things get trickier when the repeating digits don't start repeating immediately after the decimal. For example, if you have  $0.16666666\dots = 0.1\overline{6}$ , then

the 1 that appears after the decimal is not involved in the repetitions. We need to have all repeating digits start repeating right after the decimal for the above trick to work. If the decimal is followed by one digit that doesn't repeat, then 10 times the number will start repeating immediately:

$$10 \times 0.1\bar{6} = 1.\bar{6}$$

If you have four non-repeating digits after the decimal, than 10,000 times that number will start repeating right after the decimal:

$$10,000 \times 0.1111\bar{6} = 1111.\bar{6}$$

If we multiply our decimal to turn the number into a decimal we can work with, convert it into a fraction, and then divide by the multiple of 10 we originally multiplied by (10 or 10,000 above) then we can convert our original number into a fraction. We will need to remember to convert any mixed numerals into improper fractions before we divide.

$$\begin{aligned} 10 \times 0.1\bar{6} &= 1.\bar{6} \\ 10 \times 0.1\bar{6} &= 1\frac{6}{9} \\ 10 \times 0.1\bar{6} &= 1\frac{2}{3} = \frac{5}{3} \\ 0.1\bar{6} &= \frac{5}{3} \div 10 \\ 0.1\bar{6} &= \frac{5}{30} \\ 0.1\bar{6} &= \frac{1}{6} \end{aligned}$$

Similarly,

$$\begin{aligned} 10,000 \times 0.1111\bar{6} &= 1111.\bar{6} \\ 10,000 \times 0.1111\bar{6} &= 1111\frac{6}{9} \\ 10,000 \times 0.1111\bar{6} &= 1111\frac{2}{3} = \frac{3335}{3} \\ 0.1111\bar{6} &= \frac{3335}{3} \div 10,000 \\ 0.1111\bar{6} &= \frac{3335}{30,000} \\ 0.1111\bar{6} &= \frac{667}{6000} \end{aligned}$$

With these two skills, we can now convert any repeating decimal into a fraction.

## 2 The Explanation

We use standard notation for sums. For example,

$$2^1 + 2^2 + 2^3 + \dots + 2^{10} = \sum_{i=1}^{10} 2^i$$

The  $\sum$  symbol (a capital sigma from the Greek alphabet) means we will take the sum of a series of similar terms. Each term can be written in the form  $2^i$ , where  $i$  ranges from 1 (below the  $\sum$ ) to 10 (above the  $\sum$ ). The  $\sum$  notation always means taking the number below the symbol to the number above the symbol in steps of one number at a time. The variables, starting points, and ending points may change, which is why they are included in the notation.

We start by deriving a general result for one particular type of sum. In most high school curricula, student learn about geometric sequences and series. A sequence is a series of numbers that have some sort of connection or relationship. A geometric sequence is one in which the relationship is defined by a common ratio  $r$ . So, if the sequence is 1, 2, 4, 8, 16, ..., then the common ratio is 2 because  $2 \div 1 = 2$ ,  $4 \div 2 = 2$ ,  $8 \div 4 = 2$ ,  $16 \div 8 = 2$ , .... If the common ratio is less than 1 (such as  $\frac{1}{2}$ ;) then the numbers in the sequence get smaller each time.

A series is the sum of the terms in a sequence. So, if the first four terms in a sequence are 1, 2, 4 and 8, then the first four terms in the series would be 1, 3, 7 and 15 because  $1 = 1$ ,  $1 + 2 = 3$ ,  $1 + 2 + 4 = 7$ , and  $1 + 2 + 4 + 8 = 15$ . We are almost at the stage to connect these ideas to repeating decimals.

High school students also see a formula that gives the value of any term in the series if the underlying sequence is understood. Just as some people prefer to cook with a microwave, others prefer to take the time cook from scratch. If you're a "cook from the microwave" kind of mathematician, jump ahead to the finished (and numbered) equation. If not, keep reading and we'll derive the thing from scratch.

Let's start by finding a way to describe every term in a sequence. We'll need some notation: the  $n$ th term in a sequence is denoted  $t_n$ , so the first term is  $t_1$ , the second term is  $t_2$ , and so forth. The first term can also be represented by the letter  $a$ , and usually is. The common ratio between consecutive terms is  $r$  as above, which leaves  $n$  to denote the number of terms we are into the sequence. For example, if our sequence is 1, 2, 4, 8, ... as above, then  $a = 1$  and  $r = 2$  for every term in the sequence. If  $n = 1$ , we have  $t_n = 1$ . If  $n = 2$ , then  $t_n = 2$ . If  $n = 3$ , then  $t_n = 4$ , and so forth. We need to construct a formula that gives the correct  $t_n$  when given  $a, r$ , and  $n$ .

We start by recognizing that the difference between consecutive terms is a multiplied factor of  $r$ . We need to make sure that  $t_n = a$  when  $n = 1$ , and that each increment of  $n$  by 1 introduces another factor of  $r$  so that  $t_2 = ar$ ,  $t_3 = r^2$ , and so forth. If we recall that  $r^0 = 1$  for any value of  $r$ , then we can build our formula by pure logic:

$$t_n = ar^{n-1}$$

We can denote each term in a series by  $S_n$ . This means that  $S_1 = t_1$ ,  $S_2 = t_1 + t_2$ ,  $S_3 = t_1 + t_2 + t_3$ , and so forth. We now need to find a way to calculate this. We could just do it by hand, which is not a big deal for, say,  $S_2$ . However, if we need  $S_{12,345}$ , we're going to want a more efficient technique.

We start by using the summation notation introduced at the start of this section. This gives us

$$S_n = \sum_{i=1}^n ar^{n-1}$$

It's a more compact way to write the series down, but it's not any more useful when calculating the value. The next step is the one that requires the intuitive leap not expected of high school students: we multiply by  $1 - r$  on both sides of the equation. This leads to the following:

$$(1 - r) S_n = (1 - r) \sum_{i=1}^n ar^{n-1}$$

If we apply the distributive property of multiplication<sup>1</sup>, first in multiplying out the  $(1 - r)$  and then by multiplying  $r$  through the sum on the right, we get

$$\begin{aligned} (1 - r) S_n &= \sum_{i=1}^n ar^{n-1} - r \sum_{i=1}^n ar^{n-1} \\ &= \sum_{i=1}^n ar^{n-1} - \sum_{i=1}^n ar^n \end{aligned}$$

Let's take a good look at the two summation terms on the right in that second line. Both sums have  $n$  terms, each differing only by a factor of  $r$ . The first expands to  $a + ar + ar^2 + \dots + ar^{n-1}$ , while the second expands to  $ar + ar^2 + \dots + ar^{n-1} + ar^n$ . The first term in the first expansion does not appear in the second, but every subsequent term does. Similarly, every term in the second expansion appears in the first, except for the final  $ar^n$  term.

These series are being subtracted from each other. Those two unique terms will be the only terms that do not cancel out in the subtraction. In other words,

$$(1 - r) S_n = a - ar^n = a(1 - r^n)$$

We can now isolate  $S_n$  by dividing both sides by the  $(1 - r)$  term that we added arbitrarily at the start of this calculation. The final result is this:

$$S_n = \frac{a(1 - r^n)}{(1 - r)} \tag{1}$$

This is the expression given in most high schools, or the equivalent

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

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<sup>1</sup> $a(b + c) = ab + ac$

which you can get by multiplying the fraction by  $\frac{-1}{-1}$ .

We need only two more results before we can truly explain why the above method works to convert repeating decimals into fractions. We need to find out what happens if the sequence never ends. If  $n$  goes to infinity, we may have a problem.  $r^n$  gets larger and larger any time  $r > 1$ . If  $r = 1$ , then our entire process is invalidated by the division by  $r - 1$ . (However, if  $r = 1$ , then every term in the sequence is identical and our series becomes  $na$ , which is easier to deal with.) If  $0 < r < 1$ , then we have something to work with: if  $r$  is between 0 and 1 then  $r^n$  gets closer and closer to 0 as  $n$  increases, until we can use the advanced calculus based technique called “taking the limit” to say that  $\lim_{n \rightarrow \infty} r^n = 0$ . In practical terms, when the sequence never ends and  $0 < r < 1$ , we can just simplify this to  $r^n = 0$ . With this, we have

$$S_{\infty} = \frac{a}{(1-r)}$$

Our final result is for the special case where  $a = r = \frac{1}{x}$ , where  $x$  is any number greater than 1. If we substitute these values into the above expression, we get

$$S_{\infty} = \frac{\frac{1}{x}}{(1-\frac{1}{x})} = \frac{1}{x-1}$$

Now we learn why all this stuff about infinite, unending series is useful. Let’s go back to our repeating decimals, and recall the meanings of the different place values. We are used to breaking things up with place values one digit at a time, so that we have the following:

$$0.\overline{34} = 0.343434\dots = \frac{3}{10} + \frac{4}{100} + \frac{3}{1000} + \frac{4}{10000} + \dots$$

Instead, however, we can break them up into larger blocks according to the portion that repeats:

$$0.\overline{34} = \frac{34}{100} + \frac{34}{10000} + \frac{34}{1000000} + \dots = \frac{34}{100} + \frac{34}{100^2} + \frac{34}{100^3} + \dots$$

We can write this repeating decimal as a multiple of an infinite series where  $a = r = \frac{1}{100}$ :

$$0.\overline{34} = 34 \sum_{i=1}^{\infty} \left(\frac{1}{100}\right) \left(\frac{1}{100}\right)^{i-1}$$

Applying our above result for the special case of  $a = r = \frac{1}{x}$ , we see that

$$0.\overline{34} = 34 \frac{1}{100-1} = \frac{34}{99}$$

As *every* repeating fraction can be expressed as the numerator multiplied by an infinite series where  $a = r = \frac{1}{10^m}$  where  $m$  is the number of digits that repeat, we can apply the result in all cases, and the above algorithm works.