Common and Uncommon Standard Number Sets

W. Blaine Dowler

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Abstract

There are a number of important (and interesting unimportant) sets in mathematics. Sixteen of those sets are detailed here.

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1 Natural Numbers

The first set of numbers to be defined is the set of natural numbers. The set is usually labeled $\mathbb{N}$. This set is the smallest set that satisfies the following two conditions:

1. 1 is a natural number, usually written compactly as $1 \in \mathbb{N}$.
2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

As this is the smallest set that satisfies these conditions, it is constructed by starting with 1, adding 1 to 1, adding 1 to that, and so forth, such that

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Note that set theorists will often include 0 in the natural numbers. The set of natural numbers was defined formally starting with 1 long before set theorists developed a rigorous way to define numbers as sets. In that formalism, it makes more sense to start with 0, but 1 is the more common standard because it long predates modern set theory.

More advanced mathematicians may have encountered "proof by induction." This is a method of completing proofs. The proof that this method of doing proofs is valid is based on matching logical statements up with the natural number system.

Every natural number is also whole, an integer, rational, algebraic, real, complex and quaternionic.

2 Whole Numbers

The set of whole numbers, denoted $\mathbb{W}$, is defined as the union of 0 (or the set containing 0) with the natural numbers. As such, it is represented as

$$\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$$

Every whole number is also an integer, rational, algebraic, real, complex and quaternionic.

3 Prime Numbers

The prime numbers are the whole numbers which have exactly two positive whole number divisors. The set is then

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$$

They are often first defined for students as numbers which are only divisible by themselves and one, but that leads to an ambiguity: is 1 a prime number? There are a number of theorems that apply to all prime numbers, but which would *not* apply if 1 were included. For example, the prime factorization of a number is unique up to reordering. For example, $30 = 2 \times 3 \times 5 = 3 \times 2 \times 5$ etc. However, if 1 were prime, then this would no longer be the case. Instead of having the same three factors in different orders, we would have $30 = 2 \times 3 \times 5 = 1 \times 2 \times 3 \times 5 = 1^2 \times 2 \times 3 \times 5 = 1^3 \times 2 \times 3 \times 5 = 1^{500} \times 2 \times 3 \times 5$ and so forth. Making 1 an exception to almost every theorem involving prime numbers in this manner gets cumbersome. It is far more logical to define the group as the set of numbers with exactly two factors, which conforms to the common definition while neatly excluding 1.

Every prime number is also whole, an integer, rational, algebraic, real, complex and quaternionic.

4 Composite Numbers

The composite numbers are those whole numbers which have more than two divisors. While they have no consistent symbol, the set is $\{0, 4, 6, 8, 9, 10, \ldots\}$. Note that 0 is divisible by everything, and that 1 is not in this set. 1 is neither prime nor composite, and as it is the only whole number which does not belong in either of those sets, it receives no special label for this distinction.

Every composite number is also whole, an integer, rational, algebraic, real, complex and quaternionic.

5 Perfect Numbers

The perfect numbers are those numbers whose factors (aside from themselves) add up to the number. The smallest perfect number is 6. The factors of 6 are 1, 2, 3 and 6. If we take 6 itself off the list, we are left with 1, 2 and 3. Well,

$1+2+3=6$, so 6 is a perfect number. These numbers are well spread out; the first four perfect numbers are 6, 28, 496 and 8128.

Every perfect number is also natural, whole, composite, an integer, rational, algebraic, real, complex and quaternionic.

6 Integers

The integers are the numbers defined as the whole numbers and the set of additive inverses (negatives) of the whole numbers. The entire professional mathematical community uses the label $\mathbb{Z}$ for the set, after the German word Zahlen. However, many North American high schools use $\mathbb{I}$ instead, dating back to the second World War, when it was decided that students should be distanced from German influence, so a more intuitive symbol was used. The academic community did not follow suit, especially since $\mathbb{I}$ was a symbol reserved for another set entirely. As a result, many North American high school students receive quite a shock when they first enter post-secondary mathematics and lose marks for using the wrong symbol when it is the only symbol they have ever known.

The set of integers is

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

Every integer is also rational, algebraic, real, complex and quaternionic.

7 Rational Numbers

The rational numbers are any numbers which can be represented as a ratio of the form

$$q = \frac{z}{n}$$

where q is a rational number, z is an integer and n is a natural number. It is often defined as the ratio of two integers, but that definition requires the explicit exclusion of the zero denominator. The definition presented here can be used to represent any rational number without need for that exception. If $n = 1$, then we can see that the set of rational numbers includes all integers, and by extension, all whole and natural numbers. The standard symbol is $\mathbb{Q}$, because $\mathbb{R}$ was assigned to something else and there was no expectation to find a set of numbers whose name began with the letter q. Because the rational numbers include any numbers that can be written in fractional form, they include those whose decimal representations either terminate or repeat. Non-terminating, non-repeating decimals are not rational numbers.

Every rational number is also algebraic, real, complex and quaternionic.

8 Algebraic Numbers

The algebraic numbers are all of the numbers which can be the roots of a polynomial[1] with integer coefficients. This includes all rationals, since $q = \frac{z}{n}$ is the root of the simple polynomial $nx - z$. This also includes some irrationals, including (but not limited to) all simple radical numbers. (The number $\sqrt[n]{a}$ is a root of polynomial $x^n - a$.) Somewhat counterintuitively, there are algebraic numbers which can *only* be expressed in terms of the polynomial(s) for whom they act as roots. This is a side effect of the fact that there is no general algorithm which can be applied to find all roots of an arbitrary polynomial of degree five or higher. (i.e. there is no method which will lead you to the roots of $ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ for every possible value of a, b, c, d, e and f.)

Every algebraic number is also real, complex and quaternionic.

9 Real Numbers

The set of real numbers, denoted $\mathbb{R}$, is defined formally in relation to the rational numbers, as the "closure" of that set. Informally, the real numbers are any numbers you can put on a one-dimensional number line. All sets of numbers described so far will fit on such a number line, but they do not fill the number line. For example, $\sqrt{2}$ fits on the number line, but it is not rational, though it is algebraic. π is neither rational nor algebraic, but it does fit on the number line.

The formal definition, as a "closure" of another set, in this case the rationals, means you put all of your rational numbers on the number line, and close the gaps between them by filling in all of the extra numbers. A formal definition of a closure comes from real analysis, which will be the topic of Bureau 42's summer school in 2012.

Every real number is also complex and quaternionic.

[1] The roots of a polynomial in variable x are the values of x for which the polynomial equals 0.

10 Irrational Numbers

The set of irrational numbers, denoted $\overline{\mathbb{Q}}$, is also defined in relation to the rational numbers. Note that the symbol for the irrational numbers is the symbol for the rational numbers with a line over it. In mathematical logic terms, a line over a variable means "not that variable." In this case, the symbol for the irrational numbers literally means "not rational" numbers. That is how they are defined: the irrational numbers are the real numbers which are not rational, so these are the numbers which you add to the number line to close the gaps in the rational numbers when forming the real numbers. These include all non-terminating, non-repeating decimals. There are common examples, such as $\sqrt{2}$ and π, as well as less common examples, such as $0.101001000100001000001\ldots$, which has a definite pattern, but which is *not* an exact repetition.

Interestingly, there are *far* more irrational numbers than rational numbers. Both sets are infinitely large, but there are relative sizes of infinity, and the infinity that corresponds to the size of this set is much larger than the infinity that corresponds to the other sets.

Every irrational number is also real, complex and quaternionic.

11 Transcendental Numbers

The transcendental numbers are similar to the irrational numbers in that they are defined in relation to another subset of the real numbers. The transcendental numbers are those you need to close the gaps between the *algebraic* numbers on the number line. This set, then, is a subset of the irrational numbers, as it includes all irrational numbers which do not act as roots to polynomials with integer coefficients.

Every transcendental number is also irrational, real, complex and quaternionic.

12 Normal Numbers

The normal numbers may be the numbers with the least intuitive name on the list. These are also irrational numbers, but they are those whose non-repeating decimal representations show an equal distribution of values. In other words, when writing out the decimal representation, any single digit is just as likely to appear as any other digit. Likewise, any pair of digits is just as likely as any other pair, and so forth. This must be true in every base; the irrational number represented as $0.101001000100001000001\ldots$ in binary is $0.641632557\ldots$ in our normal decimal representation. This appears to have a random digit distribution in base 10 (which we are used to), but not in binary (base 2, which most of us are not used to.) For example, in binary, two consecutive 1s will not appear. Therefore, this is not a normal number.

It is exceedingly difficult to prove that any given number is normal. It is believed, though not proven, that $\sqrt{2}$, e and π are all normal numbers. If a number truly is normal, then it has an infinitely long binary representation, and this representation is equally likely to contain any finite string of digits as any other finite string. Thus, it is believed (but also not proven) that a normal number represented in binary would contain any and every possible finite string of 0s and 1s in it somewhere. In other words, if we understand normal numbers as well as we think we do, then every normal number contains anything and everything that could possibly be stored on a computer's hard drive, including the complete works of Shakespeare, your favorite movie in high definition format, and the most popular video game of the year 3025. There's no guarantee that there are a finite number of digits preceding such strings, which means you may not necessarily be able to ever find them, but they're probably in there somewhere.

Every normal number is also irrational, real, complex and quaternionic.

13 Schizophrenic Numbers

The schizophrenic numbers are the numbers which initially appear to be rational for some finite number of decimal places, but eventually reveal their irrational character. For example, $3.3333333333 + \frac{\pi}{10^{10}}$ is schizophrenic, as it appears to repeat for the first ten decimal places before revealing its irrational character.

Every schizophrenic number is also real, irrational, complex and quaternionic.

14 Imaginary Numbers

The set of imaginary numbers, which is the set that is correctly labeled $\mathbb{I}$, is the set formed by the square roots of negative numbers. Beginning with the definition[2] $\sqrt{-1} = i$, we have $\sqrt{-9} = 3i$, $\sqrt{-64} = 8i$, and so forth.

It was this set which inadvertently led to the naming of the real numbers. When first proposed, many mathematicians found imaginary numbers difficult to accept. (They are well accepted now, and have actually been measured in laboratory situations.) Thus, the real and imaginary numbers were actually named as a form of professional mocking: "you can work with your imaginary numbers if you like, but *I* work with *real* numbers." Imaginary numbers have their own number line, which is perpendicular to the usual number line, and intersects the usual number line at $0 = 0i$.

Every imaginary number is also complex and quaternionic.

15 Complex Numbers

Complex numbers are the combinations of real and imaginary numbers. For example, $3 + \sqrt{-4} = 3 + 2i$ is a complex number. If a complex number $x + yi$ is the root of a polynomial with integer coefficients, then $x - yi$ is also a root of that polynomial. $x - yi$ is the "complex conjugate" of $x + yi$, and

$$(x + yi)(x - yi) = x^2 + xyi - xyi - y^2 i^2 = x^2 - y^2 \sqrt{-1}^2 = x^2 + y^2$$

is a real number.

Every complex number is also quaternionic.

16 Quaternions

The set of quaternions is denoted $\mathbb{H}$, because $\mathbb{Q}$ was taken and they were discovered by Sir William Rowan Hamilton, so the label is the first letter of his last name. These are very much like complex numbers, but instead of having one number which squares to -1 (i), there are three: $i^2 = j^2 = k^2 = -1$. Strangely, i, j and k anticommute under multiplication, meaning $ij = -ji$, and so forth. Furthermore, $ij = k$, $jk = i$ and $ki = j$, such that $ijk = -1$. These counterintuitive numbers have applications to physics through special relativity, and (if you set the appropriate coefficients of i, j and k to zero) you will find that this set includes every set of numbers listed in this document.

[2] Note that electrical engineers were already using i for a current when they starting applying imaginary numbers to their work. In that field, $\sqrt{-1} = j$.

These sixteen are not the only sets of numbers by any means, but they are some of the most important and/or interesting sets of numbers out there.